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Finitely Continuous Hamel Functions

Abstract

A function $h\colon \mathbb{R}^n \to \mathbb{R}^k$ is called a Hamel function if it is a Hamel basis for $\mathbb{R}^{n+k}$. We prove that there exists a Hamel function which is finitely continuous (its graph can be covered by finitely many partial continuous functions). This answers the question posted in [KP].

We consider functions with values in $\mathbb{R}^k$. No distinction is made between a function and its graph. Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ be a function and $\kappa \le \mathfrak{c}$ be a cardinal number. We say that the function f is a Hamel function if f, considered as a subset of $\mathbb{R}^{n+k}$, is a Hamel basis for $\mathbb{R}^{n+k}$. The function f is called κ-continuous if it can be covered by the union of κ many partial continuous functions from $\mathbb{R}^n$. We write $f|A$ for the restriction of f to a set $A \subseteq \mathbb{R}^n$. For $B \subset \mathbb{R}^n$, the symbol $\mathrm{Lin}_{\mathbb{Q}}(B)$ stands for the smallest linear subspace of $\mathbb{R}^n$ over $\mathbb{Q}$ that contains B.

In [KP], it was asked whether there exists a Hamel function which is ω-continuous (Problem 3.2). We give an affirmative answer to this question.

Theorem 1. *There exists a Hamel function $h : \mathbb{R}^n \to \mathbb{R}^k$ which is $(n+2)$-continuous $(k, n \ge 1)$.*

Let us mention here that it is unknown whether the number $(n + 2)$ is optimal, however it cannot be replaced by 1 (e.g. $(n + 2)$-continuity cannot be replaced by continuity; see [KP, Fact 3.1 (iii)]).

Problem 2 Is the number $(n + 2)$ in Theorem 1 optimal?

Key Words: finitely continuous function; Hamel function.
Mathematical Reviews subject classification: 26A15

To prove Theorem 1, we will need the following lemma.

Lemma 3 *Let $H \subseteq \mathbb{R}^n$ be a Hamel basis. Assume that $h\colon \mathbb{R}^n \to \mathbb{R}^k$ is such that $h|H \equiv 0$. Then h is a Hamel function iff $h|(\mathbb{R}^n \setminus H)$ is one-to-one and $h[\mathbb{R}^n \setminus H] \subseteq \mathbb{R}^k$ is a Hamel basis.*

PROOF. First assume that h is a Hamel function. We will show that $h|(\mathbb{R}^n \setminus H)$ is a bijection onto a Hamel basis. Let $y \in \mathbb{R}^k$. There exist $x_1, \ldots, x_j \in \mathbb{R}^n$ and $q_1, \ldots q_j \in \mathbb{Q}$ such that $\sum_1^j q_i h(x_i) = y$. But since $h|H \equiv 0$ we get $y = \sum_1^j q_i h(x_i) = \sum_{x_i \notin H} q_i h(x_i)$. Hence $\mathrm{Lin}_{\mathbb{Q}}(h[\mathbb{R}^n \setminus H]) = \mathbb{R}^k$.

Next suppose that $\sum_1^l p_i h(x_i) = 0$ for some distinct $x_1, \ldots, x_l \in (\mathbb{R}^n \setminus H)$ and $p_1, \ldots p_l \in \mathbb{Q}$. Since $H \subseteq \mathbb{R}^n$ is a Hamel basis, there exist $x_{l+1}, \ldots, x_m \in H$ and $p_{l+1}, \ldots, p_m \in \mathbb{Q}$ such that $\sum_{l+1}^m p_i x_i = -\sum_1^l p_i x_i$. Recall that $h|H \equiv 0$, hence $\sum_1^m p_i(x_i, h(x_i)) = (0, 0)$. Since h is a Hamel function we conclude that $p_i = 0$ for all $i = 1, \ldots, m$. This finishes the proof that $h|(\mathbb{R}^n \setminus H)$ is a bijection onto a Hamel basis.

Now we prove the converse. To see that h is a Hamel function, first observe that the graph of h is linearly independent over $\mathbb{Q}$. Indeed, let $\sum_1^r q_i\,(x_i, h(x_i)) = 0$ for some $x_1, \ldots, x_r \in \mathbb{R}^n$ and $q_1, \ldots q_r \in \mathbb{Q}$. Then

$$\begin{aligned}\sum_1^r q_i\,(x_i, h(x_i)) = \sum_{x_i \in H} q_i\,(x_i, h(x_i)) + \sum_{x_i \notin H} q_i\,(x_i, h(x_i)) &= \\ \sum_{x_i \in H} q_i\,(x_i, 0) + \sum_{x_i \notin H} q_i\,(x_i, h(x_i)) &= 0.\end{aligned}$$

Hence $\sum_{x_i \notin H} q_i h(x_i) = 0$. Since $h|(\mathbb{R}^n \setminus H)$ is a bijection onto a Hamel basis, we conclude that $q_i = 0$ for $x_i \notin H$. Consequently, $\sum_{x_i \in H} q_i x_i = 0$. This implies that $q_i = 0$ for $x_i \in H$.

To see that $\mathrm{Lin}_{\mathbb{Q}}(h) = \mathbb{R}^{n+k}$, choose $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$. Since $h[\mathbb{R}^n \setminus H]$ is a Hamel basis for $\mathbb{R}^k$, there exist $x_1, \ldots, x_s \in \mathbb{R}^n$ and $p_1, \ldots, p_s \in \mathbb{Q}$ such that $\sum_1^s p_i h(x_i) = y$. Similarly, since H is a Hamel basis for $\mathbb{R}^n$, there exist $x_{s+1}, \ldots, x_t \in H \subseteq \mathbb{R}^n$ and $p_{s+1}, \ldots, p_t \in \mathbb{Q}$ such that $\sum_{s+1}^t p_i x_i = x - \sum_1^s p_i x_i$. Next observe that $\sum_1^t p_i h(x_i) = \sum_1^s p_i h(x_i) = y$ by the assumption $h|H \equiv 0$. Finally, we obtain $\sum_1^t p_i(x_i, h(x_i)) = (x, y)$. So $\mathrm{Lin}_{\mathbb{Q}}(h) = \mathbb{R}^{n+k}$. ■

PROOF OF THEOREM 1. Let $P \subseteq \{(x, 0, \ldots, 0) \in \mathbb{R}^k \colon x \notin \mathbb{Q}\}$ be a perfect set linearly independent over $\mathbb{Q}$ (see e.g., [MK, Theorem 2, p. 270]) and $Y \subseteq (\mathbb{R} \setminus \mathbb{Q})^k$ be Hamel basis such that $P \subseteq Y$. The existence of such a basis follows from the fact that $\mathrm{Lin}_{\mathbb{Q}}((\mathbb{R} \setminus \mathbb{Q})^k) = \mathbb{R}^k$ and the fact from elementary linear algebra that every linearly independent set can be extended to a linear basis. Next choose a Hamel basis $H \subseteq (\mathbb{R}\setminus\{0\})\times\mathbb{R}\times\cdots\times\mathbb{R} \subseteq \mathbb{R}^n$ such that H is dense in $\mathbb{R}^n$ (such a basis exists because $\mathrm{Lin}_{\mathbb{Q}}((\mathbb{R}\setminus\{0\})\times\mathbb{R}\times\cdots\times\mathbb{R}) = \mathbb{R}^n$). Since $X = \mathbb{R}^n \setminus H$ has topological dimension $\leq (n-1)$ (as the complement of a dense set; see [HW, Theorem IV.3 p. 44]), it can be decomposed into n 0-dimensional spaces $E_1, \ldots, E_n$ (see [HW, Theorem III.3 p. 32]). For every perfect set $Q \subseteq \mathbb{R}$ and 0-dimensional space E, there exists an embedding $g\colon E \to Q$. (See e.g., [HW, Theorem V.6 p. 65].) Hence, if $P = P_1 \cup P_2 \cup \cdots \cup P_n$ is a partition of P into n perfect sets, then there exists an embedding $g_{P_i}^{E_i}\colon E_i \to P_i$ for every $i \leq n$. Now define $g_1 = \bigcup_1^n g_{P_i}^{E_i}\colon X \to Y$ and note that it is an injective n-continuous function. Next, since Y is also 0-dimensional (as a subset of a 0-dimensional space $(\mathbb{R}\setminus\mathbb{Q})^k$), it can be embedded into any perfect set, hence also into the set X. Let $g_2\colon Y \to X$ be an embedding. Now, following the proof of Cantor-Bernstein Theorem, define a function $f\colon X \to Y$ by

$$f(x) = \begin{cases} g_1(x) & \text{if } x \notin A \\ g_2^{-1}(x) & \text{if } x \in A, \end{cases}$$

where $A_0 = g_2[Y \setminus g_1[X]]$, $A_{m+1} = g_2[g_1[A_m]]$ for $m \geq 0$, and $A = \bigcup_{m=0}^{\infty} A_m$. The function f is a bijection. To see this observe that $g_1[X\setminus A] = g_1[X]\setminus g_1[A]$ and

$$\begin{aligned} g_2^{-1}[A] &= \bigcup_{m=0}^{\infty} g_2^{-1}[A_m] = (Y \setminus g_1[X]) \cup \bigcup_{m=0}^{\infty} g_1[A_m] \\ &= (Y \setminus g_1[X]) \cup g_1[A]. \end{aligned}$$

Hence $g_1[X \setminus A] \cap g_2^{-1}[A] = \emptyset$ and $g_1[X \setminus A] \cup g_2^{-1}[A] = Y$. Since both g_1 and g_2^{-1} are injections, the latter implies that f is bijective.

Now, by recalling that g_1 is n-continuous and g_2^{-1} is continuous, we conclude that f is $(n+1)$-continuous. Finally, we define $h\colon \mathbb{R}^n \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0 & \text{if } x \in H \\ f(x) & \text{if } x \notin H. \end{cases}$$

It follows from Lemma 3 that h is a Hamel function. Obviously, h is $(n+2)$-continuous. ∎

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