To the memory of my Mother

ON FUNCTIONS WHOSE GRAPH IS A HAMEL BASIS II

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Abstract. We say that a function $h\colon \mathbb{R} \to \mathbb{R}$ is a Hamel function ($h \in \mathrm{HF}$) if h, considered as a subset of $\mathbb{R}^2$, is a Hamel basis for $\mathbb{R}^2$. We show that $\mathrm{A}(\mathrm{HF}) \geq \omega$, i.e., for every finite $F \subseteq \mathbb{R}^{\mathbb{R}}$ there exists $f \in \mathbb{R}^{\mathbb{R}}$ such that $f + F \subseteq \mathrm{HF}$. From the previous work of the author it then follows that $\mathrm{A}(\mathrm{HF}) = \omega$ (see [P]).

The terminology is standard and follows [C]. The symbols $\mathbb{R}$ and $\mathbb{Q}$ stand for the sets of all real and all rational numbers, respectively. A basis of $\mathbb{R}^n$ as a linear space over $\mathbb{Q}$ is called Hamel basis. For $Y \subset \mathbb{R}^n$, the symbol $\mathrm{Lin}_{\mathbb{Q}}(Y)$ stands for the smallest linear subspace of $\mathbb{R}^n$ over $\mathbb{Q}$ that contains Y. The zero element of $\mathbb{R}^n$ is denoted by 0. All the linear algebra concepts are considered for the field of rational numbers. The cardinality of a set X we denote by $|X|$. In particular, $\mathfrak{c}$ stands for $|\mathbb{R}|$. Given a cardinal κ, we let $\mathrm{cf}(\kappa)$ denote the cofinality of κ. We say that a cardinal κ is regular if $\mathrm{cf}(\kappa) = \kappa$. For any set X, the symbol $[X]^{\kappa}$ denotes the set $\{Z \subseteq X : |Z| < \kappa\}$. For $A, B \subseteq \mathbb{R}^n$, $A + B$ stands for $\{a + b : a \in A, b \in B\}$.

We consider only real-valued functions. No distinction is made between a function and its graph. For any two partial real functions f, g we write $f + g$, $f - g$ for the sum and difference functions defined on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. The class of all functions from a set X into a set Y is denoted by Y^X. We write $f|A$ for the restriction of $f \in Y^X$ to the set $A \subseteq X$. For $B \subseteq \mathbb{R}^n$ its characteristic function is denoted by χ_B. For any function $g \in \mathbb{R}^X$ and any family of functions $F \subseteq \mathbb{R}^X$ we define $g + F = \{g + f : f \in F\}$. For any planar set P, we denote its x-projection by $\mathrm{dom}(P)$.

The cardinal function $\mathrm{A}(\mathrm{F})$, for $\mathrm{F} \subsetneq \mathbb{R}^X$, is defined as the smallest cardinality of a family $G \subseteq \mathbb{R}^X$ for which there is no $g \in \mathbb{R}^X$ such that $g + G \subseteq \mathrm{F}$. Recall that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a Hamel function ($f \in \mathrm{HF}(\mathbb{R}^n)$) if f, considered as a subset of $\mathbb{R}^{n+1}$, is a Hamel basis for $\mathbb{R}^{n+1}$. In [P], it was proved that $3 \leq \mathrm{A}(\mathrm{HF}(\mathbb{R}^n)) \leq \omega$. In the same paper, the author asked whether $\mathrm{A}(\mathrm{HF}(\mathbb{R}^n)) = \omega$ (Problem 3.5). The following theorem gives a positive answer to this question.

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Theorem 1. $\mathrm{A}(\mathrm{HF}(\mathbb{R}^n)) \geq \omega$, *i.e. for every finite* $F \subseteq \mathbb{R}^{\mathbb{R}^n}$, *there exists* $g \in \mathbb{R}^{\mathbb{R}^n}$ *such that* $g + F \subseteq \mathrm{HF}(\mathbb{R}^n)$.

Before we prove the theorem we state and prove the following lemmas.

Lemma 2. *Let* $b_1, \ldots, b_m \in \mathbb{R}$ *be arbitrary numbers. There exists a linear basis* C *of* $\mathrm{Lin}_{\mathbb{Q}}(b_1, \ldots, b_m)$ *such that* $b_i + C$ *is also a basis of* $\mathrm{Lin}_{\mathbb{Q}}(b_1, \ldots, b_m)$, *for every* $i \leq m$.

Proof. Without loss of generality we may assume that $\mathrm{Lin}_{\mathbb{Q}}(b_1, \ldots, b_m) \neq \{0\}$. Let $C' = \{{c_1}', \ldots, {c_k}'\}$ be any linear basis of $\mathrm{Lin}_{\mathbb{Q}}(b_1, \ldots, b_n)$. So, for every $i \leq m$ there are ${p_{i1}}', \ldots, {p_{ik}}' \in \mathbb{Q}$ such that

$$\sum_j {p_{ij}}' {c_j}' = b_i.$$

Now, choose $q \in \mathbb{Q} \setminus \{0\}$ satisfying the following condition for all i

$$q \sum_j {p_{ij}}' \neq -1.$$

We claim that $C = \{c_1, \ldots, c_k\} = \frac{1}{q} C' = \{\frac{1}{q}{c_1}', \ldots, \frac{1}{q}{c_k}'\}$ is the desired basis. To prove this we need to show that for every $i \leq m$

- $b_i + C$ is linearly independent.

  To see this consider a zero linear combination $\sum_j p_{ij}(b_i + c_j) = 0$. We have that $\sum_j p_{ij} c_j = -b_i \sum_j p_{ij}$. If $\sum_j p_{ij} = 0$ then obviously $p_{i1} = \cdots = p_{ik} = 0$. So we may assume that $\sum_j p_{ij} \neq 0$. Next we divide both sides of $\sum_j p_{ij} c_j = -b_i \sum_j p_{ij}$ by $-\sum_j p_{ij}$ and obtain that $\sum_j \frac{p_{ij}}{-\sum_j p_{ij}} c_j = b_i$. On the other hand $\sum_j {p_{ij}}' {c_j}' = \sum_j {p_{ij}}' q c_j = b_i$. So we conclude that $\frac{p_{ij}}{-\sum_j p_{ij}} = q {p_{ij}}'$ for all $j \leq k$ and consequently $q \sum_j {p_{ij}}' = \sum_j \frac{p_{ij}}{-\sum_j p_{ij}} = -1$. A contradiction.

Now, since $\dim(\mathrm{Lin}_{\mathbb{Q}}(b_i + C)) = \dim(\mathrm{Lin}_{\mathbb{Q}}(C))$ and $\mathrm{Lin}_{\mathbb{Q}}(b_i + C) \subseteq \mathrm{Lin}_{\mathbb{Q}}(C)$, we conclude that $\mathrm{Lin}_{\mathbb{Q}}(b_i + C) = \mathrm{Lin}_{\mathbb{Q}}(C) = \mathrm{Lin}_{\mathbb{Q}}(b_1, \ldots, b_m)$. ■

Let us note here that the above lemma cannot generalized onto infinite case. As a counterexample take $\{b_1, b_2, b_3, \ldots\} = \mathbb{Q}$ and observe that there is no basis C with the required properties.

Lemma 3. [PR, Lemma 2] *Let* $H \subseteq \mathbb{R}^n$ *be a Hamel basis. Assume that* $h\colon \mathbb{R}^n \to \mathbb{R}$ *is such that* $h|H \equiv 0$. *Then* h *is a Hamel function iff* $h|(\mathbb{R}^n \setminus H)$ *is one-to-one and* $h[\mathbb{R}^n \setminus H] \subseteq \mathbb{R}$ *is a Hamel basis.*

Lemma 4. *Let X be a set of cardinality $\mathfrak{c}$ and $k \geq 1$. TFAE:*

(a) *for all $f_1, \dots, f_k \colon \mathbb{R}^n \to \mathbb{R}$, there exists $f \in \mathbb{R}^{\mathbb{R}^n}$ such that $f + f_i \in \mathrm{HF}(\mathbb{R}^n)$ $(i = 1, \dots, k)$.*

(b) *for all $g_1, \dots, g_k \in \mathbb{R}^X$, there exists $g \in \mathbb{R}^X$ such that $g + g_i$ is one-to-one and $(g + g_i)[X] \subseteq \mathbb{R}$ is a Hamel basis $(i = 1, \dots, k)$.*

Proof. $(a) \Rightarrow (b)$ Choose a Hamel basis $H \subseteq \mathbb{R}^n$ and a bijection $p \colon \mathbb{R}^n \setminus H \to X$. Put $f_i = (g_i \circ p) \cup (0|H)$. By (a), there exists an $f \in \mathbb{R}^{\mathbb{R}^n}$ such that $f + f_i \in \mathrm{HF}(\mathbb{R}^n)$ $(i = 1, \dots, k)$. Now, let $A \in \mathrm{Add}(\mathbb{R}^n)$ be such that $f|H = A|H$ and put $f' = f - A$. Note that $f' + f_i = (f + f_i) - A \in \mathrm{HF}(\mathbb{R}^n) - \mathrm{Add}(\mathbb{R}^n) = \mathrm{HF}(\mathbb{R}^n)$ (see [P, Fact 3.1]) and also $(f' + f_i)|H \equiv 0, (i = 1, \dots, k)$. Hence, by Lemma 3 we claim that $(f' + f_i)|(\mathbb{R}^n \setminus H)$ is a bijection onto a Hamel basis. Now define $g = f' \circ p^{-1}$ and note that it is the required function.

$(b) \Rightarrow (a)$ Let H be like above. Choose $A_i \in \mathrm{Add}(\mathbb{R}^n)$ such that $f_i|H \equiv A_i|H$ for every $i = 1, \dots, k$. Put $X = \mathbb{R}^n \setminus H$ and $g_i = (f_i - A_i)|X$ for $i = 1, \dots, k$. By (b), there exists a $g \colon X \to \mathbb{R}$ such that $g + g_i$ is a bijection between X and a Hamel basis. Define $f = g \cup (0|H)$ and observe that $f + f_i = [f + (f_i - A_i)] + A_i = [(g + g_i) \cup (0|H)] + A_i$. Since $(g + g_i) \cup (0|H) \in \mathrm{HF}(\mathbb{R}^n)$ by Lemma 3, using [P, Fact 3.1] we conclude that $[(g + g_i) \cup (0|H)] + A_i \in \mathrm{HF}(\mathbb{R}^n)$ for each $i = 1, \dots, k$. Hence f is the required function. ∎

Lemma 5. *Let X be a set of cardinality $\mathfrak{c}$, $\omega \leq \kappa < \mathfrak{c}$, and $f_1, \dots f_k \in \mathbb{R}^X$ be functions such that $|f_i[X]| = \mathfrak{c}$. Then there exist pairwise disjoint subsets $A_1, \dots A_n \subseteq X$ of cardinality κ^+ each and satisfying the following property: for every $i = 1, \dots, k$ and $j = 1, \dots, n$ the restriction $f_i|A_j$ is one-to-one or constant and $|f_i[\bigcup A_j]| = \kappa^+$ (i.e. f_i is one-to-one on at least one of the sets).*

Proof. We prove the lemma by induction on k. If $k = 1$, then the conclusion is obvious (note that $\kappa^+ \leq \mathfrak{c}$). Now assume that the lemma holds for $k \in \omega$ and let $f_1, \dots f_{k+1} \in \mathbb{R}^X$ be functions such that $|f_i[X]| = \mathfrak{c}$. Based on the inductive assumption, let $A_1, \dots A_n \subseteq X$ be sets with the required properties for the functions $f_1, \dots f_k$. If $|f_{k+1}[\bigcup A_i]| = \kappa^+$, then by reducing the original sets $A_1, \dots A_n$ we will obtain sets which work for all the functions $f_1, \dots f_{k+1}$. In the case when $|f_{k+1}[\bigcup A_i]| \leq \kappa$, we can find a subset $A_{n+1} \subseteq X$ disjoint with $\bigcup_1^n A_i$ such that $|A_{n+1}| = \kappa^+$ and $f_{k+1}|A_{n+1}$ is injective. Now, by appropriately reducing the sets $A_1, \dots A_{n+1}$ we will obtain the desired sets. ∎

Lemma 6. *Let X be a set of cardinality $\mathfrak{c}$, $f_1, \dots f_k \in \mathbb{R}^X$ be functions such that $|f_i[X]| = \mathfrak{c}$, $B_0, B_1 \subseteq \mathbb{R}$ be such that $|B_0 \cup B_1| < \mathfrak{c}$, and $y \in \mathbb{R} \setminus \mathrm{Lin}_{\mathbb{Q}}(B_0)$. Then there exist $y_1, \dots, y_n \in \mathbb{R}$ and $x_1, \dots, x_n \in X$ such that*

(a) $\sum_1^n y_j = y$,

(b) $\{y_1, \dots, y_n\}, \{y_j + f_i(x_j) \colon j = 1, \dots, n\}$ *are both linearly independent over* $\mathbb{Q}$ *and* $\mathrm{Lin}_{\mathbb{Q}}(\{y_1, \dots, y_n\}) \cap \mathrm{Lin}_{\mathbb{Q}}(B_0) = \mathrm{Lin}_{\mathbb{Q}}(\{y_j + f_i(x_j) \colon j = 1, \dots, n\}) \cap \mathrm{Lin}_{\mathbb{Q}}(B_1) = \{0\}$ *for all* $i = 1, \dots, k$.

Proof. Put $\kappa = |B_0 \cup B_1 \cup \omega|$ and let $A_1, \dots A_n \subseteq X$ be the sets from Lemma 5 for functions $f_1, \dots f_k$. First we will define the values $y_1, \dots, y_n$. Let $\{b_1, \dots b_s\}$ be the set of all values such that $f_i|A_j \equiv b_l$ for some i, j, l. Choose $y_2, \dots, y_n$ to be linearly independent over $\mathbb{Q}$ such that $\mathrm{Lin}_{\mathbb{Q}}(\{y_2, \dots, y_n\}) \cap \mathrm{Lin}_{\mathbb{Q}}(B_0 \cup B_1 \cup \{b_1, \dots, b_s, y\}) = \{0\}$. This can be easily done by extending the basis of $\mathrm{Lin}_{\mathbb{Q}}(B_0 \cup B_1 \cup \{b_1, \dots, b_s, y\})$ to a Hamel basis and selecting $(n-1)$ elements from the extension. Next define $y_1 = y - (y_2 + \dots + y_n)$.

Obviously $\sum_1^n y_j = y$. We claim that $\{y_1, \dots, y_n\}$ is linearly independent over $\mathbb{Q}$ and $\mathrm{Lin}_{\mathbb{Q}}(\{y_1, \dots, y_n\}) \cap \mathrm{Lin}_{\mathbb{Q}}(B_0) = \{0\}$. Assume that $\alpha_1 y_1 + \dots + \alpha_n y_n = 0$ for some rationals $\alpha_1, \dots, \alpha_n$. From the definition of y_1 we get $(\alpha_2 - \alpha_1) y_2 + \dots + (\alpha_n - \alpha_1) y_n = -\alpha_1 y$. Based on the way $y_2, \dots, y_n$ were selected we conclude that $\alpha_1 = 0$ and consequently $\alpha_2 = \dots = \alpha_n = 0$. Next assume that $q_1 y_1 + \dots + q_n y_n = b$ for some rationals $q_1, \dots, q_n$ and $b \in \mathrm{Lin}_{\mathbb{Q}}(B_0)$. Then, proceeding similarly like above, we obtain that $(q_2 - q_1) y_2 + \dots + (q_n - q_1) y_n \in \mathrm{Lin}_{\mathbb{Q}}(B_0 \cup \{y\})$, which implies that $q_1 = \dots = q_n$. Consequently, if $q_1 \neq 0$, then we could conclude that $y \in \mathrm{Lin}_{\mathbb{Q}}(B_0)$. That would contradict one of the assumptions of the lemma. Hence $q_1 = \dots = q_n = 0$ and the sequence $y_1, \dots, y_n$ satisfies the required conditions.

Before we define the sequence $x_1, \dots, x_n$, we observe some additional properties of $y_1, \dots, y_n$. Fix $1 \le i \le k$. Let $A_{i_1}, \dots, A_{i_l}$ $(i_1 < \dots < i_l)$ be all the sets on which f_i is constant and let $b_{i_1}, \dots, b_{i_l}$ be the values of f_i on these sets, respectively. Note that properties of the sets $A_1, \dots, A_n$ imply that $\{i_1, \dots, i_l\} \subsetneq \{1, \dots, n\}$. We will show that

(1) $(y_{i_1} + b_{i_1}), \dots, (y_{i_l} + b_{i_l})$ are linearly independent,

(2) $\mathrm{Lin}_{\mathbb{Q}}(\{(y_{i_1} + b_{i_1}), \dots, (y_{i_l} + b_{i_l})\}) \cap \mathrm{Lin}_{\mathbb{Q}}(B_1) = \{0\}$.

To see (1) assume that $\alpha_1(y_{i_1} + b_{i_1}) + \dots + \alpha_l(y_{i_l} + b_{i_l}) = 0$ for some rationals $\alpha_1, \dots, \alpha_l$. This implies $\alpha_1 y_{i_1} + \dots + \alpha_l y_{i_l} = -(\alpha_1 b_{i_1} + \dots + \alpha_l b_{i_l}) \in \mathrm{Lin}_{\mathbb{Q}}(B_0 \cup B_1 \cup \{b_1, \dots, b_s, y\})$. If $i_1 \neq 1$, then it easily follows that $\alpha_1 = \dots = \alpha_l = 0$. If $i_1 = 1$, then we can write $\alpha_1 y_{i_1} + \dots + \alpha_l y_{i_l} = \alpha_1 y_1 + \alpha_2 y_{i_2} + \dots + \alpha_l y_{i_l} = \alpha_1[y - (y_2 + \dots + y_n)] + \alpha_2 y_{i_2} + \dots + \alpha_l y_{i_l} \in \mathrm{Lin}_{\mathbb{Q}}(B_0 \cup B_1 \cup \{b_1, \dots, b_s, y\})$.

Consequently, $-\alpha_1(y_2+\cdots+y_n)+\alpha_2 y_{i_2}+\cdots+\alpha_l y_{i_l} \in \mathrm{Lin}_{\mathbb{Q}}(B_0\cup B_1\cup\{b_1,\ldots,b_s,y\})$. Since $\{i_1,\ldots,i_l\} \subsetneq \{1,\ldots,n\}$, after simplifying the expression $-\alpha_1(y_2+\cdots+y_n)+\alpha_2 y_{i_2}+\cdots+\alpha_l y_{i_l}$, there will be at least one term y_j with the coefficient being exactly $-\alpha_1$. Hence, we conclude that $\alpha_1 = 0$ and as a consequence of that $\alpha_2 = \cdots = \alpha_l = 0$. This finishes the proof of (1). Similar argument shows (2).

Next we will define the elements $x_1,\ldots,x_n \in X$ (by induction). Choose

$$x_1 \in A_1 \setminus \bigcup_{i\le k, f_i \text{ is 1-1 on } A_1} f_i^{-1}[\mathrm{Lin}_{\mathbb{Q}}(B_1\cup\{b_1,b_2,\ldots,b_s,y_1,\ldots,y_n\})].$$

This choice is possible since $|A_1| = \kappa^+ > \kappa \ge |\mathrm{Lin}_{\mathbb{Q}}(B_1\cup\{b_1,b_2,\ldots,b_s,y_1,\ldots,y_n\})|$ and together with the condition (2) assures that $\mathrm{Lin}_{\mathbb{Q}}(\{y_1+f_i(x_1)\colon f_i|A_1 \text{ is 1-1}\})\cap \mathrm{Lin}_{\mathbb{Q}}(B_1\cup\{b_1,b_2,\ldots,b_s,y_1,\ldots,y_n\}) \subseteq \{0\}$ and $\mathrm{Lin}_{\mathbb{Q}}(\{y_1+f_i(x_1)\})\cap\mathrm{Lin}_{\mathbb{Q}}(B_1) = \{0\}$ for all $i \le k$.

Now assume that $x_1 \in A_1,\ldots,x_{m-1}\in A_{m-1}$ $(m<n)$ have been defined and they satisfy the following property: $(\star)$ $\{y_j+f_i(x_j)\colon j=1,\ldots,m-1\}$ is linearly independent, $\mathrm{Lin}_{\mathbb{Q}}(\{y_j+f_i(x_j)\colon j\le m-1 \text{ and } f_i|A_j \text{ is 1-1}\})\cap\mathrm{Lin}_{\mathbb{Q}}(B_1\cup\{b_1,b_2,\ldots,b_s,y_1,\ldots,y_n\}) \subseteq \{0\}$, and $\mathrm{Lin}_{\mathbb{Q}}(\{y_j+f_i(x_j)\colon j=1,\ldots,m-1\})\cap\mathrm{Lin}_{\mathbb{Q}}(B_1) = \{0\}$ for all $i=1,\ldots,k$. Choose $x_m\in A_m$ such that

$$x_m \notin \bigcup_{i\le k, f_i \text{ is 1-1 on } A_m} f_i^{-1}[\mathrm{Lin}_{\mathbb{Q}}(B_1\cup\{b_1,b_2,\ldots,b_s,y_1,\ldots,y_n,f_i(x_1),\ldots,f_i(x_{m-1})\})].$$

The choice of x_m implies that

$$y_m+f_i(x_m)\notin\mathrm{Lin}_{\mathbb{Q}}(B_1\cup\{b_1,b_2,\ldots,b_s,y_1,\ldots,y_n,f_i(x_1),\ldots,f_i(x_{m-1})\})$$

for all $i\le k$ such that f_i is 1-1 on A_m. This combined with the inductive assumption $(\star)$ and the conditions (1) and (2) leads to the conclusion that $\{y_j+f_i(x_j)\colon j=1,\ldots,m\}$ is linearly independent, $\mathrm{Lin}_{\mathbb{Q}}(\{y_j+f_i(x_j)\colon j\le m \text{ and } f_i|A_j \text{ is 1-1}\})\cap\mathrm{Lin}_{\mathbb{Q}}(B_1\cup\{b_1,b_2,\ldots,b_s,y_1,\ldots,y_n\})\subseteq\{0\}$, and $\mathrm{Lin}_{\mathbb{Q}}(\{y_j+f_i(x_j)\colon j=1,\ldots,m\})\cap\mathrm{Lin}_{\mathbb{Q}}(B_1)=\{0\}$ for all $i=1,\ldots,k$. Based on the induction we claim that the sequence $x_1,\ldots,x_n\in X$ has been constructed and it satisfies the following condition: $\{y_j+f_i(x_j)\colon j=1,\ldots,n\}$ is linearly independent and $\mathrm{Lin}_{\mathbb{Q}}(\{y_j+f_i(x_j)\colon j=1,\ldots,n\})\cap\mathrm{Lin}_{\mathbb{Q}}(B_1)=\{0\}$ for all $i=1,\ldots,k$.

Summarizing, the sequences $x_1,\ldots,x_n\in X$ and $y_1,\ldots,y_n\in\mathbb{R}$ satisfying the conditions (a) and (b) have been constructed. ■

Remark 7. Let $A'\subseteq A$ and $f_1,f_2\colon A\to\mathbb{R}$. If $(f_1-f_2)[A]\subseteq\mathrm{Lin}_{\mathbb{Q}}(f_1[A'])\cap\mathrm{Lin}_{\mathbb{Q}}(f_2[A'])$, then $\mathrm{Lin}_{\mathbb{Q}}(f_1[A])=\mathrm{Lin}_{\mathbb{Q}}(f_2[A])$.

The remark easily follows from the equality

$$\sum_{1}^{l} \alpha_i f_1(x_i) = \sum_{1}^{l} \alpha_i f_2(x_i) + \sum_{1}^{l} \alpha_i [f_1(x_i) - f_2(x_i)].$$

Proof of Theorem 1. Let X be a set of cardinality $\mathfrak{c}$. By Lemma 4, it suffices to show that for arbitrary $f_1, \ldots, f_k \colon X \to \mathbb{R}$ there exists a function $g \colon X \to \mathbb{R}$ such that $g + f_i$ is 1-1 and $(g + f_i)[X]$ is a Hamel basis ($i = 1, \ldots, k$). The proof in the general case will be by transfinite induction with the use of the previously stated auxiliary results. However, in the special case when $|f_i[X]| < \mathfrak{c}$ for all i, it can be presented without the use of induction. The method is interesting and also used in part of the proof of general case, so we present it here. Assume that $|f_i[X]| < \mathfrak{c}$ for all i, let $V = \mathrm{Lin}_{\mathbb{Q}}(\bigcup f_i[X])$, and $\lambda < \mathfrak{c}$ be the cardinality of a linear basis of V. Choose $Z \subseteq X$ such that $|Z| = \lambda$ and $f_i|Z$ is a constant function for every i and let $\{b_1, \ldots, b_m\} = \bigcup f_i[Z]$. Next we define a Hamel basis H. Let C be a basis of $\mathrm{Lin}_{\mathbb{Q}}(b_1, \ldots, b_m)$ from Lemma 2, H_1 be an extension of C to a basis of V, and finally H be an extension of H_1 to a Hamel basis. Define $g \colon X \to H$ as a bijection with the property that $g[Z] = H_1$. We claim that $g + f_i$ is 1-1 and $(g + f_i)[X]$ is a Hamel basis ($i = 1, \ldots, k$). To see this recall that $b_j + C$ is linearly independent, $\mathrm{Lin}_{\mathbb{Q}}(b_j + C) = \mathrm{Lin}_{\mathbb{Q}}(C) = \mathrm{Lin}_{\mathbb{Q}}(\{b_1, \ldots, b_m\})$ (see Lemma 2), and $C \subseteq H_1$. This implies that $\mathrm{Lin}_{\mathbb{Q}}(b_j + H_1) = \mathrm{Lin}_{\mathbb{Q}}(H_1)$, $b_j + (H_1 \setminus C)$ is linearly independent, and as a consequence, $b_j + H_1$ is linearly independent. Therefore, since $f_i[Z] = \{b_j\}$ for some j, we have that $(g + f_i)[Z] = b_j + H_1$. Thus $(g + f_i)[Z]$ is linearly independent and $\mathrm{Lin}_{\mathbb{Q}}((g + f_i)[Z]) = \mathrm{Lin}_{\mathbb{Q}}(H_1)$. Finally, since $f_i[X] \subseteq \mathrm{Lin}_{\mathbb{Q}}(H_1) = \mathrm{Lin}_{\mathbb{Q}}((g + f_i)[Z])$, we can similarly conclude that $(g + f_i)[X]$ is linearly independent and $\mathrm{Lin}_{\mathbb{Q}}((g + f_i)[X]) = \mathrm{Lin}_{\mathbb{Q}}(g[X]) = \mathrm{Lin}_{\mathbb{Q}}(g[X]) = \mathbb{R}$. This finishes the proof of the special case.

Now we prove the result for arbitrary functions $f_1, \ldots, f_k \colon X \to \mathbb{R}$. We start by dividing $\{f_1, \ldots, f_k\}$ into abstract classes according to the relation: $f_i \approx f_j$ *iff* $|(f_i - f_j)[X]| < \mathfrak{c}$ (it is easy to verify that this is an equivalence relation). Put $K = \bigcup_i \bigcup_{f_j \approx f_i} (f_i - f_j)[X]$, $\kappa = |\omega \cup K|$, and note that $\kappa < \mathfrak{c}$. There exists a set $Z \subseteq X$ such that $|Z| = \kappa^+$ and for all i, j the function $(f_i - f_j)|Z$ is one-to-one or constant (the existence of such a set can be shown by using an argument similar to the one from the proof of Lemma 5; obviously, if $f_i \approx f_j$, then $(f_i - f_j)|Z$ is constant). Our goal is to define $g \colon Z' \to \mathbb{R}$ for some $Z' \subseteq Z$ such that for every $i \leq k$ $g + f_i$ is injective, $(g + f_i)[Z']$ is linearly independent, and $K \subseteq \mathrm{Lin}_{\mathbb{Q}}((g + f_i)[Z'])$.

Define $V = \mathrm{Lin}_{\mathbb{Q}}(K)$ and introduce another equivalence relation among the functions $f_1, \ldots, f_k$: $f_i \cong f_j$ *iff* $(f_i - f_j)|Z$ is constant. Note that $\approx \subseteq \cong$. Let

$f_{i_1},\dots,f_{i_l}$ be representatives of the abstract classes of the relation $\cong$. Consider $\bigcup_{s=1}^{l}\bigcup_{f_j\cong f_{i_s}}(f_j - f_{i_s})[Z] = \{b_1,\dots,b_m\}$. By Lemma 2, there exists a linear basis C of $\mathrm{Lin}_{\mathbb{Q}}(\{b_1,\dots,b_m\})$ such that $b_r + C$ ($r \le m$) is also a linear basis for $\mathrm{Lin}_{\mathbb{Q}}(\{b_1,\dots,b_m\})$. Let H_1 be a linear basis of V extending C. Choose a set $Z_1 \subseteq Z$ such that $|Z_1| = |H_1|$ and $(f_j - f_{i_1})[Z_1]$ is linearly independent and $\mathrm{Lin}_{\mathbb{Q}}((f_j - f_{i_1})[Z_1]) \cap V = \{0\}$ for all $f_j \ncong f_{i_1}$. This can be done since $|Z| = \kappa^+ > |V| \ge |H_1|$ and $(f_j - f_{i_1})|Z$ is injective for every $f_j \ncong f_{i_1}$. Let $g_1'\colon Z_1 \to H_1$ be a bijection and define $g\colon Z_1 \to \mathbb{R}$ by $g = g_1' - f_{i_1}$. Then $g + f_j$ is one-to-one for all j, $(g+f_j)[Z_1]$ is linearly independent for all j, $\mathrm{Lin}_{\mathbb{Q}}((g+f_j)[Z_1]) = V$ for $f_j \cong f_{i_1}$ (see the argument in the special case in the beginning of the proof), and $\mathrm{Lin}_{\mathbb{Q}}((g+f_j)[Z_1]) \cap V = \{0\}$ for $f_j \ncong f_{i_1}$ (the latter follows from the fact that if Y_1 and Y_2 are both linearly independent and $\mathrm{Lin}_{\mathbb{Q}}(Y_1)\cap\mathrm{Lin}_{\mathbb{Q}}(Y_2) = \{0\}$, then Y_1+Y_2 is also linearly independent and $\mathrm{Lin}_{\mathbb{Q}}(Y_1)\cap\mathrm{Lin}_{\mathbb{Q}}(Y_1+Y_2) = \mathrm{Lin}_{\mathbb{Q}}(Y_1)\cap\mathrm{Lin}_{\mathbb{Q}}(Y_1+Y_2) = \{0\}$) .

Next choose a set $Z_2 \subseteq Z \setminus Z_1$ such that $|Z_2| = |H_1|$, $(f_j - f_{i_2})[Z_2]$ is linearly independent, and $\mathrm{Lin}_{\mathbb{Q}}((f_j - f_{i_2})[Z_2]) \cap \mathrm{Lin}_{\mathbb{Q}}(\bigcup_1^k (g+f_i)[Z_1]) = \{0\}$ for all $f_j \ncong f_{i_2}$ (note that $V \subseteq \bigcup_1^k (g+f_i)[Z_1]$ since $\mathrm{Lin}_{\mathbb{Q}}((g+f_{i_1})[Z_1]) = V$). This choice is possible for similar reasons as in the case of Z_1. Let $g_2'\colon Z_2 \to H_1$ be a bijection and extend g onto $Z_1 \cup Z_2$ by defining it on Z_2 as $g = g_2' - f_{i_2}$. Then $g + f_j$ is one-to-one for all j, $(g+f_j)[Z_1 \cup Z_2]$ is linearly independent for all j, $V \subseteq \mathrm{Lin}_{\mathbb{Q}}((g+f_j)[Z_1 \cup Z_2])$ for $f_j \cong f_{i_1}$ or $f_j \cong f_{i_2}$, and $\mathrm{Lin}_{\mathbb{Q}}((g+f_j)[Z_1 \cup Z_2]) \cap V = \{0\}$ for $f_j \ncong f_{i_1}$ and $f_j \ncong f_{i_2}$.

By continuing this process (or more formally, by using the mathematical induction), we construct a sequence of pairwise disjoint sets $Z_1, Z_2, \dots, Z_l \subseteq Z$ and a partial real function $g\colon Z' \to \mathbb{R}$ ($Z' = Z_1 \cup \dots \cup Z_l$) such that for each $j = 1,\dots,k$, $g+f_j$ is one-to-one, $(g+f_j)[Z']$ is linearly independent, and $V \subseteq \mathrm{Lin}_{\mathbb{Q}}((g+f_j)[Z'])$. Observe also that $|Z'| \le \kappa$. Therefore $|X \setminus Z'| = \mathfrak{c}$.

In the following part of the proof, we will use the transfinite induction to extend the partial function g onto the whole set X making sure it possesses the desired properties. We will make use of Lemma 6 and Remark 7. First notice that if $Z' \subseteq A \subseteq X$ and $g\colon A \to \mathbb{R}$ is any extension of $g\colon Z' \to \mathbb{R}$ then for $f_j \approx f_i$ we have that $((g+f_j) - (g+f_i))[A] = (f_j - f_i)[A] \subseteq (f_j - f_i)[X] \subseteq V \subseteq \mathrm{Lin}_{\mathbb{Q}}((g + f_i)[Z']) \cap \mathrm{Lin}_{\mathbb{Q}}((g+f_j)[Z'])$. Hence the remark implies that $\mathrm{Lin}_{\mathbb{Q}}((g+f_i)[A]) = \mathrm{Lin}_{\mathbb{Q}}((g+f_j)[A])$. Thus, when extending the function g it will suffice to consider only the representatives of the abstract classes of the relation $\approx$. Let $f_{j_1},\dots,f_{j_t}$ be those functions. Let $H = \{h_\xi\colon \xi < \mathfrak{c}\}$ be a Hamel basis and $\{x_\xi\colon \xi < \mathfrak{c}\}$ be an enumeration of $X \setminus Z'$. We will define a sequence of pairwise disjoint finite sets $\{X_\xi \colon \xi < \mathfrak{c}\}$ such that $\bigcup_{\xi<\mathfrak{c}} X_\xi = X \setminus Z'$, $x_\xi \in \bigcup_{\beta\leq\xi} X_\beta$ and an extension of g onto X such that for each $\xi < \mathfrak{c}$ the following condition holds

(P_ξ) $g + f_{j_r}$ is one-to-one, $(g + f_{j_r})[Z' \cup \bigcup_{\beta\leq\xi} X_\beta]$ is linearly independent, and $h_\xi \in \mathrm{Lin}_{\mathbb{Q}}((g + f_{j_r})[Z' \cup \bigcup_{\beta\leq\xi} X_\beta])$ for all $r = 1, \ldots, t$.

Notice that this will finish the proof of our main theorem. To perform the inductive construction, fix $\alpha < \mathfrak{c}$ and assume that the sets X_ξ have been defined for all $\xi < \alpha$ and the function g extended onto $Z' \cup \bigcup_{\xi<\alpha} X_\xi$ in such a way that (P_ξ) is satisfied for each $\xi < \alpha$.

If $x_\alpha \notin Z' \cup \bigcup_{\xi<\alpha} X_\xi$, then define $g(x_\alpha) \notin \bigcup_{r=1}^{t} \mathrm{Lin}_{\mathbb{Q}}((g + f_{j_r})[Z' \cup \bigcup_{\xi<\alpha} X_\xi] \cup \{f_{j_r}(x_\alpha)\})$. This assures that $g+f_{j_r}$ is one-to-one and $(g+f_{j_r})[Z'\cup\bigcup_{\xi<\alpha} X_\xi\cup\{x_\alpha\}]$ is linearly independent $(r = 1, \ldots, t)$. Next, if $h_\alpha \in (g + f_{j_1})[Z' \cup \bigcup_{\xi<\alpha} X_\xi \cup \{x_\alpha\}]$, then put $X_{\alpha 1} = \emptyset$. Otherwise, we apply Lemma 6 to functions $f_{j_r} - f_{j_1} \colon X \setminus (Z' \cup \bigcup_{\xi<\alpha} X_\xi \cup \{x_\alpha\}) \to \mathbb{R}$ $(r = 2, \ldots, t)$, $B_0 = \mathrm{Lin}_{\mathbb{Q}}((g + f_{j_1})[Z' \cup \bigcup_{\xi<\alpha} X_\xi \cup \{x_\alpha\}])$, $B_1 = \mathrm{Lin}_{\mathbb{Q}}(\bigcup_{r=2}^{t}(g + f_{j_r})[Z' \cup \bigcup_{\xi<\alpha} X_\xi \cup \{x_\alpha\}])$, and $y = h_\alpha$. Hence there exist $y_{1j_1}, \ldots, y_{n_1 j_1} \in \mathbb{R}$ and $x_{1j_1}, \ldots, x_{n_1 j_1} \in X\setminus(Z'\cup\bigcup_{\xi<\alpha} X_\xi\cup\{x_\alpha\})$ such that the conditions (a) and (b) from the lemma are satisfied. We define $X_{\alpha 1} = \{x_{1j_1}, \ldots, x_{n_1 j_1}\}$ and $g(x_{ij_1}) = y_{ij_r} - f_{j_1}(x_{ij_1})$ $(i = 1 \ldots, n_1)$. By repeating the above steps for the other functions $f_{j_2}, \ldots, f_{j_t}$ (the sets B_0 and B_1 need to be appropriately extended in each step) we obtain pairwise disjoint sets $X_{\alpha 1}, \ldots, X_{\alpha t} \subseteq X\setminus(Z'\cup\bigcup_{\xi<\alpha} X_\xi\cup\{x_\alpha\}$ and an extension of g onto $Z' \cup \bigcup_{\xi\leq\alpha} X_\xi$ (where $X_\alpha = X_{\alpha 1} \cup \cdots \cup X_{\alpha t} \cup \{x_\alpha\}$). Observe that the conditions (a) and (b) from Lemma 6 imply that (P_α) holds. This completes the inductive construction and also the proof of Theorem 1. ∎

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