To the memory of Alexandra

On lineability and additivity of real functions with finite preimages

Krzysztof Płotka[1]

Department of Mathematics, University of Scranton, Scranton, PA 18510, USA

Abstract

We study lineability of real functions with finite preimages. In particular, we prove that the class of n-to-one functions contains a vector subspace of dimension n but not of dimension $(n+1)$. Additionally, we give examples of star-like families of functions (closed under multiplication by a non-zero scalar) for which lineability is less than additivity.

Keywords: lineability, additivity, n-to-one functions, Hamel functions
2010 MSC: 15A03, 26A21, 03E75

1. Introduction

The symbols $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of positive integers, rational and real numbers, respectively. The cardinality of a set X is denoted by the symbol $|X|$. In particular, $|\mathbb{N}|$ is denoted by ω and $|\mathbb{R}|$ is denoted by $\mathfrak{c}$. We consider only real-valued functions. No distinction is made between a function and its graph. We write $f|A$ for the restriction of f to the set $A \subseteq \mathbb{R}$. The symbol χ_A denotes the characteristic function of the set A. For any subset Y of a vector space V and any $v \in V$ we define $v+Y = \{v+y\colon y \in Y\}$.

The problem of finding a "large" vector subspace contained in a given subset of a vector space has gained on importance in recent years and significant

Email address: krzysztof.plotkak@scranton.edu (Krzysztof Płotka)

[1]The work was supported in part by the 2014 intersession research grant from the University of Scranton.

number of articles have been written on the topic (see [1–4, 6–13]). More specifically, many well known families of functions (considered as subsets of the vector space $\mathbb{R}^{\mathbb{R}}$) have been studied in that context. We will recall here some of the most recent definitions related to the topic (see [1, 2, 6]). Let $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$, $E \subseteq \mathbb{R}$ be a field, and κ be a cardinal number. We say that $\mathcal{F}$ is *κ-lineable over E* if $\mathcal{F} \cup \{0\}$ contains a subspace of $\mathbb{R}^{\mathbb{R}}$ (considered as a space over E) of dimension κ. The *(coefficient of) lineability of the family $\mathcal{F}$ over the field E* is denoted by $\mathcal{L}_E(\mathcal{F})$ and defined as follows

$$\mathcal{L}_E(\mathcal{F}) = \min\{\kappa\colon \mathcal{F} \text{ is not } \kappa\text{-lineable over } E\}.$$

In the case $E = \mathbb{R}$ we simply write $\mathcal{L}(\mathcal{F})$.

In this paper we focus on the lineability of functions with finite preimages. Let us recall the definitions of the classes of functions considered in the article. A function $f\colon \mathbb{R} \to \mathbb{R}$ is a:

- *linearly independent* function if the graph of f is a linearly independent subset of $\mathbb{R}^2$ (over $\mathbb{Q}$) ($f \in \mathrm{LIF}$);
- *Hamel* function if the graph of f is a Hamel basis for $\mathbb{R}^2$ ($f \in \mathrm{HF}$);
- *n-to-one* function ($n \geq 1$) if for every $y \in \mathbb{R}$, $|f^{-1}(y)| = n$ or 0 ($f \in \mathrm{F}_n$);
- *finite-to-one* function if for every $y \in \mathbb{R}$, $|f^{-1}(y)| < \omega$ ($f \in \mathrm{F}_{<\omega}$).

In addition, we introduce the symbol $\mathrm{F}_{<n}$ ($n \geq 2$) to denote the family of functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $y \in \mathbb{R}$, $|f^{-1}(y)| < n$.

The cardinal function $\mathrm{A}(\mathcal{F})$, for $\mathcal{F} \subsetneq \mathbb{R}^X$, is defined as the smallest cardinality of a family $\mathcal{G} \subseteq \mathbb{R}^X$ for which there is no $g \in \mathbb{R}^X$ such that $g + \mathcal{G} \subseteq \mathcal{F}$ (see [15]). It was investigated for various classes of real functions. The following remark gives the values of $\mathrm{A}(\mathrm{F}_n)$, $\mathrm{A}(\mathrm{F}_{<n})$, and $\mathrm{A}(\mathrm{F}_{<\omega})$.

Remark 1.1. $\mathrm{A}(\mathrm{F}_1) = \mathrm{A}(\mathrm{F}_{<n}) = \mathrm{A}(\mathrm{F}_{<\omega}) = \mathfrak{c}$ and $\mathrm{A}(\mathrm{F}_n) = 2$ for $n \geq 2$.

Proof. Let $\mathcal{F} = \{f \in \mathbb{R}^{\mathbb{R}}\colon f|(\mathbb{R} \setminus \mathbb{N}) \equiv 0\}$. Note that $|\mathcal{F}| = \mathfrak{c}$ and $g + \mathcal{F} \nsubseteq \mathrm{F}_{<\omega}$ for every real function g since $-g\chi_{\mathbb{N}} \in \mathcal{F}$ and $g + (-g\chi_{\mathbb{N}})$ is constant on $\mathbb{N}$. Hence $\mathrm{A}(\mathrm{F}_1) \leq \mathrm{A}(\mathrm{F}_{<n}) \leq \mathrm{A}(\mathrm{F}_{<\omega}) \leq \mathfrak{c}$. To show that $\mathrm{A}(\mathrm{F}_1) \geq \mathfrak{c}$ let $\mathcal{H} \subseteq \mathbb{R}^{\mathbb{R}}$ be such that $|\mathcal{H}| < \mathfrak{c}$. Let $g \in \mathbb{R}^{\mathbb{R}}$ be such that for all $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$ we have

$$(g(x_1) + \{h(x_1)\colon h \in \mathcal{H}\}) \cap (g(x_2) + \{h(x_2)\colon h \in \mathcal{H}\}) = \emptyset$$

(such a function g can easily constructed using transfinite induction). Notice that $g + \mathcal{H} \subseteq \mathrm{F}_1$.

To see that $\mathrm{A}(\mathrm{F}_n) = 2$ for $n \geq 2$, recall first that $\mathrm{A}(\mathcal{F}) \geq 2$ for every non-empty family $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$. To justify the inequality $\mathrm{A}(\mathrm{F}_n) \leq 2$ define $f_1 \equiv 0$ and $f_2 = \chi_{\{0\}}$ and note that for every function g we have $g + \{f_1, f_2\} \nsubseteq \mathrm{F}_n$. □

Gámez-Merino, Muñoz-Fernández, and Seoane-Sepúlveda (see [8]) established a connection between the two cardinal functions A and $\mathcal{L}$. Namely, they proved the following theorem.

Theorem 1.2. [8, Theorem 2.4] *If $\mathcal{F} \subsetneq \mathbb{R}^{\mathbb{R}}$ is star-like (i.e., $c\mathcal{F} \subseteq \mathcal{F}$ for every $c \in \mathbb{R} \setminus \{0\}$) and $\mathrm{A}(\mathcal{F}) > \mathfrak{c}$, then $\mathcal{L}(\mathcal{F}) > \mathrm{A}(\mathcal{F})$.*

The above theorem was generalized by Bartoszewicz and Głąb [2, Theorem 2.2]: *If $\mathcal{F} \subsetneq \mathbb{R}^{\mathbb{R}}$ is star-like, $E \subseteq \mathbb{R}$ is an infinite field, and $A(\mathcal{F}) > |E|$, then $\mathcal{L}_E(\mathcal{F}) > A(\mathcal{F})$.* Theorem 1.2 guarantees that families of functions with large additivity (greater than $\mathfrak{c}$) contain a subspace of large dimension (greater than or equal to additivity). The authors asked a question whether the above theorem can be extended to classes of functions with lower additivity. Specifically, does Theorem 1.2 remain true if $2 < \mathrm{A}(\mathcal{F}) \leq \mathfrak{c}$? A negative answer to this question was given by Bartoszewicz and Głąb in [2]. In particular, they proved that for every $\kappa \leq \mathfrak{c}$ there exists a family $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ such that $\mathrm{A}(\mathcal{F}) = \kappa$ and $\mathcal{L}(\mathcal{F}) = 2$. It still may be of interest to find such families among those that were previously defined and studied in other contexts. We identify three of such classes of functions.

Remark 1.3. The families $\mathrm{HF}, \mathrm{LIF}$, and F_1 are all star-like. In addition, $\mathrm{A}(\mathrm{HF}) = \omega$, $\mathrm{A}(F_1) = \mathrm{A}(\mathrm{LIF}) = \mathfrak{c}$ and $\mathcal{L}(\mathrm{HF}) = \mathcal{L}(F_1) = \mathcal{L}(\mathrm{LIF}) = 2$.

Proof. The equalities $\mathrm{A}(\mathrm{HF}) = \omega$ and $\mathrm{A}(\mathrm{LIF}) = \mathfrak{c}$ were proved in [16, 17]. For $\mathrm{A}(F_1) = \mathfrak{c}$ see Remark 1.1 and for $\mathcal{L}(F_1) = 2$ see Theorem 2.3. (See also [9].) To see $\mathcal{L}(\mathrm{HF}) = \mathcal{L}(\mathrm{LIF}) = 2$ note that for any two functions f, g we have $(g(0)f - f(0)g)(0) = 0$, hence $g(0)f - f(0)g \notin \mathrm{LIF}$. The fact that the classes $\mathrm{HF}, \mathrm{LIF}$, and F_1 are star-like easily follows from the definitions of these classes. □

2. Main results

We first determine $\mathcal{L}_{\mathbb{Q}}(\mathrm{HF})$ and $\mathcal{L}_{\mathbb{Q}}(\mathrm{LIF})$. To do that we will make use of the following lemma.

Lemma 2.1. [18, Lemma 3] *Let $B \subseteq \mathbb{R}$ be a Hamel basis. Assume that $h\colon \mathbb{R} \to \mathbb{R}$ is such that $h|B \equiv 0$. Then h is a Hamel function iff $h|(\mathbb{R} \setminus B)$ is one-to-one and $h[\mathbb{R} \setminus B] \subseteq \mathbb{R}$ is a Hamel basis.*

Theorem 2.2. $\mathcal{L}_{\mathbb{Q}}(\mathrm{HF}) = \mathcal{L}_{\mathbb{Q}}(\mathrm{LIF}) = \mathfrak{c}^+$.

Proof. First observe that since $\mathrm{HF} \subseteq \mathrm{LIF}$ we have $\mathcal{L}_{\mathbb{Q}}(\mathrm{HF}) \leq \mathcal{L}_{\mathbb{Q}}(\mathrm{LIF})$. Therefore it suffices to show $\mathcal{L}_{\mathbb{Q}}(\mathrm{LIF}) \leq \mathfrak{c}^+$ and $\mathcal{L}_{\mathbb{Q}}(\mathrm{HF}) \geq \mathfrak{c}^+$. The inequality $\mathcal{L}_{\mathbb{Q}}(\mathrm{LIF}) \leq \mathfrak{c}^+$ follows from the fact that in any collection of functions of cardinality $> \mathfrak{c}$, there are two distinct functions f_1, f_2 such that $f_1(0) = f_2(0)$ and then $f_1 - f_2 \notin \mathrm{LIF}$. (Note here that the inequality $\mathcal{L}_{\mathbb{Q}}(\mathrm{LIF}) \geq \mathfrak{c}^+$ follows from theorem of Bartoszewicz and Głąb [2, Theorem 2.2].)

By Lemma 2.1, the inequality $\mathcal{L}_{\mathbb{Q}}(\mathrm{HF}) \geq \mathfrak{c}^+$ follows from the existence of a family of Hamel bases $H_\alpha = \{h^\alpha_\xi : \xi < \mathfrak{c}\} \subseteq \mathbb{R}$ ($\alpha < \mathfrak{c}$) such that

$$(\Delta)\ \{p_1 h^{\alpha_1}_\xi + \cdots + p_k h^{\alpha_k}_\xi : \xi < \mathfrak{c}\} \text{ is a Hamel basis}$$

for all $\alpha_1 < \cdots < \alpha_k < \mathfrak{c}$ and $p_1, \ldots, p_k \in \mathbb{Q} \setminus \{0\}$, $k \geq 1$. Indeed, if such a family exists then functions defined by

$$f_\alpha = (B \times \{0\}) \cup \{(x_\xi, h^\alpha_\xi)\colon \xi < \mathfrak{c}\}$$

(where B is a Hamel basis and $\mathbb{R} \setminus B = \{x_\xi\colon \xi < \mathfrak{c}\}$) are linearly independent over $\mathbb{Q}$ and $\mathrm{span}_{\mathbb{Q}}\{f_\alpha\colon \alpha < \mathfrak{c}\} \subseteq \mathrm{HF} \cup \{0\}$.

Let $V = \{H_\alpha\}$ be a maximal set of Hamel bases with the property (Δ) and assume that $|V| < \mathfrak{c}$. Using transfinite induction we will define a Hamel basis $H = \{h_\xi\colon \xi < \mathfrak{c}\}$ such that $V \cup \{H\}$ still possesses the property (Δ). Let $\mathbb{R} = \{y_\gamma\colon \gamma < \mathfrak{c}\}$ and fix $\lambda_0 < \mathfrak{c}$. Assume that the construction has been carried out for every $\lambda < \lambda_0$ satisfying the following conditions:

(i) $\lambda \in I_\lambda$, where $I_\lambda = \{\xi < \mathfrak{c}\colon h_\xi$ is defined after stage $\lambda\}$,

(ii) $|I_\lambda| \leq \max(\omega, \lambda, |V|)$,

(iii) for all $k \geq 1$, $\alpha_1 < \cdots < \alpha_k < |V|$, and $p_1, \ldots, p_k \in \mathbb{Q}$ we have that $\{h_\xi + p_1 h^{\alpha_1}_\xi + \cdots + p_k h^{\alpha_k}_\xi : \xi \in I_\lambda\}$ is linearly independent over $\mathbb{Q}$ and $y_\lambda \in \mathrm{span}_{\mathbb{Q}}\{h_\xi + p_1 h^{\alpha_1}_\xi + \cdots + p_k h^{\alpha_k}_\xi : \xi \in I_\lambda\}$.

Put $I = \{\xi < \mathfrak{c}\colon h_\xi$ is defined before the step $\lambda_0\} = \bigcup_{\lambda<\lambda_0} I_\lambda$ and observe that $|I| \leq \max(\omega, \lambda_0, |V|)$. If $\lambda_0 \notin I$ then choose

$$h_{\lambda_0} \notin \mathrm{span}_{\mathbb{Q}}(\{h_\xi\colon \xi \in I\} \cup \{h^\kappa_\xi\colon \xi \in I, \kappa < |V|\}).$$

This can be done because the cardinality of this span is not bigger than $\max(\omega, \lambda_0, |V|) < \mathfrak{c}$. Hence the condition (i) is satisfied for λ_0.

Next we will assure that the condition (iii) holds for λ_0 using an additional induction process. Fix a well ordering

$$\{(k^\tau, \alpha_1^\tau, \ldots, \alpha_{k^\tau}^\tau, p_1^\tau, \ldots, p_{k^\tau}^\tau) \colon \tau < \max(\omega, |V|)\} \text{ of}$$

$$\{(k, \alpha_1, \ldots \alpha_k, p_1 \ldots, p_k) \colon k \geq 1, \alpha_1 < \cdots < \alpha_k < |V|, p_1, \ldots, p_k \in \mathbb{Q}\}$$

and an ordinal number $\beta < \max(\omega, |V|)$. Suppose that the induction process has been performed for every $\tau < \beta$. Let $I^\tau = \{\xi < \mathfrak{c} \colon h_\xi \text{ is defined after step } \tau\}$ (observe here that $I \cup \{\lambda_0\} \subseteq I^0$) and assume that

(a) $|I^\tau| \leq \max(\omega, \lambda_0, |V|)$,

(b) for all $k \geq 1$, $\alpha_1 < \cdots < \alpha_k < |V|$, $p_1, \ldots, p_k \in \mathbb{Q}$ we have that $\{h_\xi + p_1 h_\xi^{\alpha_1} + \cdots + p_k h_\xi^{\alpha_k} : \xi \in I^\tau\}$ is linearly independent over $\mathbb{Q}$,

(c) $y_{\lambda_0} \in \operatorname{span}_{\mathbb{Q}}\{h_\xi + p_1^\tau h_\xi^{\alpha_1^\tau} + \cdots + p_{k^\tau}^\tau h_\xi^{\alpha_{k^\tau}^\tau} : \xi \in I^\tau\}$.

We need

$$y_{\lambda_0} \in \operatorname{span}_{\mathbb{Q}}\{h_\xi + p_1^\beta h_\xi^{\alpha_1^\beta} + \cdots + p_{k^\beta}^\beta h_\xi^{\alpha_{k^\beta}^\beta} : \xi \in \bigcup_{\tau<\beta} I^\tau\}.$$

If this is not the case, choose $\gamma \in \mathfrak{c} \setminus \bigcup_{\tau<\beta} I^\tau$ such that for all $m \geq 1$, $p_1, \ldots, p_m \in \mathbb{Q} \setminus \{0\}$, and $\alpha_1 < \cdots < \alpha_m < |V|$ we have that $p_1 h_\gamma^{\alpha_1} + \cdots + p_m h_\gamma^{\alpha_m}$ is not an element of

$$\operatorname{span}_{\mathbb{Q}}(\{h_\xi \colon \xi \in \bigcup_{\tau<\beta} I^\tau\} \cup \{h_\xi^\kappa \colon \xi \in \bigcup_{\tau<\beta} I^\tau, \kappa < |V|\} \cup \{y_{\lambda_0}\}).$$

Such a γ exists as the above span has cardinality less than $\mathfrak{c}$ and by the condition (Δ) the set $\{p_1 h_\xi^{\alpha_1} + \cdots + p_m h_\xi^{\alpha_m} \colon \xi < \mathfrak{c}\}$ is a Hamel basis for all $m \geq 1$, $p_1, \ldots, p_m \in \mathbb{Q} \setminus \{0\}$, and $\alpha_1 < \cdots < \alpha_m < |V|$.

Set $h_\gamma = y_{\lambda_0} - (p_1^\beta h_\gamma^{\alpha_1^\beta} + \cdots + p_{k^\beta}^\beta h_\gamma^{\alpha_{k^\beta}^\beta})$ and $I^\beta = \bigcup_{\tau<\beta} I^\tau \cup \{\gamma\}$. Obviously $|I^\beta| \leq |\bigcup_{\tau<\beta} I^\tau| + 1 \leq \max(\omega, \lambda_0, |V|)$ and

$$y_{\lambda_0} \in \operatorname{span}_{\mathbb{Q}}\{h_\xi + p_1^\beta h_\xi^{\alpha_1^\beta} + \cdots + p_{k^\beta}^\beta h_\xi^{\alpha_{k^\beta}^\beta} : \xi \in I^\beta\}.$$

Now, to verify that condition (b) holds for β suppose that for some $k \geq 1$, $\alpha_1 < \dots < \alpha_k < |V|$, $p_1, \dots, p_k \in \mathbb{Q}$, $\{h_\xi + p_1 h_\xi^{\alpha_1} + \dots + p_k h_\xi^{\alpha_k} : \xi \in I^\beta\}$ are linearly dependent over $\mathbb{Q}$. Then $h_\gamma + p_1 h_\gamma^{\alpha_1} + \dots + p_k h_\gamma^{\alpha_k}$ would be in

$$\operatorname{span}_{\mathbb{Q}}(\{h_\xi \colon \xi \in \bigcup_{\tau<\beta} I^\tau\} \cup \{h_\xi^\kappa \colon \xi \in \bigcup_{\tau<\beta} I^\tau, \kappa < |V|\})$$

which in turn would imply that $p_1 h_\gamma^{\alpha_1} + \dots + p_k h_\gamma^{\alpha_k} - (p_1^\beta h_\gamma^{\alpha_1^\beta} + \dots + p_{k^\beta}^\beta h_\gamma^{\alpha_{k^\beta}^\beta})$ is in

$$\operatorname{span}_{\mathbb{Q}}(\{h_\xi \colon \xi \in \bigcup_{\tau<\beta} I^\tau\} \cup \{h_\xi^\kappa \colon \xi \in \bigcup_{\tau<\beta} I^\tau, \kappa < |V|\} \cup \{y_{\lambda_0}\}).$$

Based on the way γ was selected, the latter could only happen if (1) $p_1 = \dots = p_k = p_1^\beta = \dots = p_{k^\beta}^\beta = 0$ or (2) $k = k^\beta$, $\alpha_i = \alpha_i^\beta$, and $p_i = p_i^\beta (i = 1, \dots, k)$. If (1) was true then $y_{\lambda_0} \in \operatorname{span}_{\mathbb{Q}}\{h_\xi \colon \xi \in \bigcup_{\tau<\beta} I^\tau\}$ which we assumed was not the case when defining h_γ. If (2) was true then $y_{\lambda_0} = h_\gamma + p_1^\beta h_\gamma^{\alpha_1^\beta} + \dots + p_{k^\beta}^\beta h_\gamma^{\alpha_{k^\beta}^\beta}$ would be in

$$\operatorname{span}_{\mathbb{Q}}(\{h_\xi + p_1^\beta h_\xi^{\alpha_1^\beta} + \dots + p_{k^\beta}^\beta h_\xi^{\alpha_{k^\beta}^\beta} : \xi \in \bigcup_{\tau<\beta} I^\tau\})$$

which, again, would result in a contradiction.

Hence, we can assume that the above induction process has been carried out for all $\beta < \max(\omega, |V|)$ and consequently that the condition (iii) holds for λ_0. Note that $I_{\lambda_0} = \bigcup_{\beta<\max(\omega,|V|)} I^\beta$ and therefore $|I_{\lambda_0}| \leq \max(\omega, \lambda_0, |V|)$. This completes the step λ_0 of the definition of H. It follows from the condition (iii) of the inductive construction that H is a Hamel basis (use $p_1 = \dots = p_k = 0$) and that $V \cup \{H\}$ possesses the property (Δ). This contradicts the assumption that $V = \{H_\alpha\}$ is a maximal family of Hamel bases satisfying (Δ). Hence we conclude that $|V| = \mathfrak{c}$. □

The following theorem gives the lineability of the functions with finite preimages (F_n and $F_{<\omega}$).

Theorem 2.3.

(i) $\mathcal{L}(\mathrm{F}_n) = n + 1$ *and* $\mathcal{L}_{\mathbb{Q}}(\mathrm{F}_n) = \mathfrak{c}^+$ *for* $n \geq 1$.

(ii) $\mathcal{L}(\mathrm{F}_{<\omega}) = \mathcal{L}_{\mathbb{Q}}(\mathrm{F}_{<\omega}) = \mathfrak{c}^+$.

In the proof of the above theorem we will use the following lemma.

Lemma 2.4. *Let $\zeta < \mathfrak{c}$ and $\mathcal{E}_m (m \geq 1)$ be a collection of $(m-1)$-dimensional subspaces of $\mathbb{R}^m$ such that $|\mathcal{E}_m| < \mathfrak{c}$ and $v_E \in \mathbb{R}^m$ for $E \in \mathcal{E}_m$. Then there exists $(y_0, y_1, \dots) \in (\mathbb{R} \setminus \{0\})^\zeta$ such that for every $m \geq 1$ and all $\xi_1 < \dots < \xi_m < \zeta$ we have $(y_{\xi_1}, \dots, y_{\xi_m}) \notin \bigcup_{E \in \mathcal{E}_m} (v_E + E)$.*

Proof. Choose $y'_0 \in \mathbb{R} \setminus \{0\}$. Next pick $\gamma < \zeta$ and assume that y'_ξ is defined for every $\xi < \gamma$ and that the sequence $(y'_0, y'_1, \dots) \in (\mathbb{R} \setminus \{0\})^\gamma$ has the following property:

($\star$) for every $k \geq 2$ and all $\xi_1 < \dots < \xi_k < \gamma$ we have that for every $m \geq k$ and $E \in \mathcal{E}_m$ if $\mathbb{R}^k \times \{0\}^{m-k} \not\subseteq E$ then $(y'_{\xi_1}, \dots, y'_{\xi_k}, \underbrace{0, \dots, 0}_{(m-k)\ 0's}) \notin E$.

Now fix $m \geq k \geq 2$, $E \in \mathcal{E}_m$, and $\xi_1 < \dots < \xi_{k-1} < \gamma$. Assume that $\mathbb{R}^k \times \{0\}^{m-k} \not\subseteq E$. We claim that there is at most one $y \in \mathbb{R}$ such that $(y'_{\xi_1}, \dots, y'_{\xi_{k-1}}, y, \underbrace{0, \dots, 0}_{(m-k)\ 0's}) \in E$. If that was not the case then we would have that $(\underbrace{0, \dots, 0}_{(k-1)\ 0's}, 1, \underbrace{0, \dots, 0}_{(m-k)\ 0's}) \in E$ and consequently

$$(y'_{\xi_1}, \dots, y'_{\xi_{k-1}}, \underbrace{0, \dots, 0}_{(m-k+1)\ 0's}) \in E.$$

If $k = 2$ then the latter would imply that $\mathbb{R}^k \times \{0\}^{m-k} \subseteq E$. If $k \geq 3$ then using the inductive assumption ($\star$) we would conclude that $\mathbb{R}^{k-1} \times \{0\}^{m-k+1} \subseteq E$, which in combination with $(\underbrace{0, \dots, 0}_{(k-1)\ 0's}, 1, \underbrace{0, \dots, 0}_{(m-k)\ 0's}) \in E$ would imply again that $\mathbb{R}^k \times \{0\}^{m-k} \subseteq E$. In either case ($k = 2$ or $k \geq 3$) we would get a contradiction with our assumption about E. Let us denote the y from above by $y_{k,m,\xi_1,\dots,\xi_{k-1},E}$ (if the y doesn't exist then we can set $y_{k,m,\xi_1,\dots,\xi_{k-1},E} = 0$). Now choose y'_γ to be a non-zero element of

$$\mathbb{R} \setminus \bigcup_{k \leq m, \xi_1 < \dots < \xi_{k-1}, E} \{y_{k,n,\xi_1,\dots,\xi_{k-1},E}\}.$$

One can easily observe that the constructed sequence $(y'_0, y'_1, \dots) \in (\mathbb{R} \setminus \{0\})^\zeta$ satisfies the following property: for every $m \geq 1$ and all $\xi_1 < \dots < \xi_m < \alpha$ we have $(y'_{\xi_1}, \dots, y'_{\xi_m}) \notin \bigcup_{E \in \mathcal{E}_m} E$.

Now pick $E \in \mathcal{E}_m$ and $\xi_1 < \cdots < \xi_m < \alpha$. There exists at most one $c_{\xi_1,\ldots,\xi_m,E} \in \mathbb{R} \setminus \{0\}$ such that

$$c_{\xi_1,\ldots,\xi_m,E}(y'_{\xi_1}, \ldots, y'_{\xi_n}) \in v_E + E.$$

Choose c to be a non-zero element of $\mathbb{R}\setminus\bigcup_{m,\xi_1<\cdots<\xi_m,E}\{c_{\xi_1,\ldots,\xi_m,E}\}$ and observe that $(y_0, y_1, \ldots) = c(y'_0, y'_1, \ldots)$ has the desired property. $\square$

Proof of Theorem 2.3.

(i) To prove the inequality $\mathcal{L}(F_n) \le n+1$, let $f_1, f_2, \ldots, f_{n+1} \in \mathbb{R}^{\mathbb{R}}$ and $x_1 < x_2 < \cdots < x_{n+1} \in \mathbb{R}$. Consider the following homogeneous system of n linear equations with $(n+1)$ unknowns $a_1, a_2, \ldots, a_{n+1}$:

$$\begin{aligned} a_1 f_1(x_1) + \cdots + a_{n+1} f_{n+1}(x_1) &= a_1 f_1(x_2) + \cdots + a_{n+1} f_{n+1}(x_2) \\ &\vdots \\ a_1 f_1(x_n) + \cdots + a_{n+1} f_{n+1}(x_n) &= a_1 f_1(x_{n+1}) + \cdots + a_{n+1} f_{n+1}(x_{n+1}) \end{aligned}$$

There exists a non-trivial solution $(a_1, a_2, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$ to the above system. Hence, if $f_1, \ldots, f_{n+1}$ are linearly independent then $\operatorname{span}\{f_1, \ldots, f_{n+1}\} \not\subseteq \mathrm{F}_{<(n+1)} \cup \{0\}$.

The inequality $\mathcal{L}(F_n) \ge n+1$ is obvious for $n=1$ so we can assume that $n \ge 2$. We will define $f_1, \ldots, f_n \in \mathrm{F}_n$ on $\mathbb{R} = \{x_\xi \colon \xi < \mathfrak{c}\}$ by induction on ξ such that $f_1, \ldots, f_n$ are linearly independent and $\operatorname{span}\{f_1, \ldots, f_n\} \subseteq \mathrm{F}_n \cup \{0\}$. We will proceed as follows. Let $\mathbb{R}^n \setminus \{(0, \ldots, 0)\} = \{(a_1^\beta, \ldots, a_n^\beta) \colon \beta < \mathfrak{c}\}$. Set $f_1(x_0), \ldots, f_n(x_0)$ arbitrarily and pick $\alpha < \mathfrak{c}$. Assume that the construction has been carried out for all $\xi < \alpha$. Let $D_\xi = \operatorname{dom}(f_1) = \cdots = \operatorname{dom}(f_n)$ after stage ξ and assume that

(a) $\{x_\gamma \colon \gamma \le \xi\} \subseteq D_\xi$, $|D_\xi| \le \max(\omega, |\xi|)$, and $D_{\xi_1} \subseteq D_\xi$ for $\xi_1 \le \xi$,

(b) for every $\beta < \mathfrak{c}$ we have that $|(a_1^\beta f_1 + \cdots + a_n^\beta f_n)^{-1}(y)| \le n$ for all $y \in \mathbb{R}$,

(c) for every $\beta \le \xi$ we have that $|(a_1^\beta f_1 + \cdots + a_n^\beta f_n)^{-1}(y)| = n$ for all $y \in (a_1^\beta f_1 + \cdots + a_n^\beta f_n)[\{x_\gamma \colon \gamma < \xi\}]$.

Put $D = \bigcup_{\xi<\alpha} D_\xi$ and define $P(A) = \{(x'_1, \ldots, x'_k) \in A^k \colon x'_i \ne x'_j \text{ for } i < j \le k, k \ge 2\}$ for any $A \subseteq \mathbb{R}$. Note that the condition (b) is equivalent to the following statement: for all $(x'_1, \ldots, x'_{n+1}) \in P(D)$ the set of vectors

$$\{(f_1(x'_i) - f_1(x'_{i+1}), \ldots, f_n(x'_i) - f_n(x'_{i+1})) \colon i \le n\}$$

is linearly independent. Indeed, the latter is equivalent to the fact that for all $(x'_1,\ldots,x'_{n+1})\in P(D)$ the following homogeneous system of linear equations

$$\begin{array}{rcl} a_1f_1(x'_1)+\cdots+a_nf_n(x'_1) &=& a_1f_1(x'_2)+\cdots+a_nf_n(x'_2)\\ &\vdots& \\ a_1f_1(x'_n)+\cdots+a_nf_n(x'_n) &=& a_1f_1(x'_{n+1})+\cdots+a_nf_n(x'_{n+1}) \end{array}$$

has only the trivial solution ($a_1=\cdots=a_n=0$) since the vectors $(f_1(x'_i)-f_1(x'_{i+1}),\ldots,f_n(x'_i)-f_n(x'_{i+1}))$ ($i\le n$) are the row vectors of the matrix of coefficients of the above system.

If $x_\alpha\notin D$ then we need to define $f_1(x_\alpha),\ldots,f_n(x_\alpha)$ preserving the condition (b). Choose

$$(f_1(x_\alpha),\ldots,f_n(x_\alpha))\in\mathbb{R}^n\setminus\bigcup_{(x'_1,\ldots,x'_n)\in P(D)}\left((f_1(x'_1),\ldots,f_n(x'_1))+E_{(x'_1,\ldots,x'_n)}\right),$$

where $E_{(x'_1,\ldots,x'_n)}=\operatorname{span}\{(f_1(x'_i)-f_1(x'_{i+1}),\ldots,f_n(x'_i)-f_n(x'_{i+1}))\colon i\le n-1\}$. The above choice is possible by Lemma 2.4 (use $\zeta=n$, $m=n$, $\mathcal{E}_n=\{E_{(x'_1,\ldots,x'_n)}\colon (x'_1,\ldots,x'_n)\in P(D)\}$, and $v_E=(f_1(x'_1),\ldots,f_n(x'_1))$ for $E=E_{(x'_1,\ldots,x'_n)}$; note that $|\mathcal{E}_n|<\mathfrak{c}$ since $|P(D)|\le\max(\omega,|\alpha|)$).

Now pick $\beta\le\alpha$ and assume that for every $\beta'<\beta$ we have

$$|(a_1^{\beta'}f_1+\cdots+a_n^{\beta'}f_n)^{-1}(y)|=n \text{ for all } y\in(a_1^{\beta'}f_1+\cdots+a_n^{\beta'}f_n)[\{x_\gamma\colon\gamma<\alpha\}].$$

We will extend the functions $f_1,\ldots,f_n$ so that

$$|(a_1^{\beta}f_1+\cdots+a_n^{\beta}f_n)^{-1}(y)|=n \text{ for all } y\in(a_1^{\beta}f_1+\cdots+a_n^{\beta}f_n)[\{x_\gamma\colon\gamma<\alpha\}].$$

Assume that $|(a_1^\beta f_1+\cdots+a_n^\beta f_n)^{-1}(y^\gamma)|=n-k_\gamma$ ($0<k_\gamma<n$) for $y^\gamma=(a_1^\beta f_1+\cdots+a_n^\beta f_n)(x_\gamma)$. Pick $(x_1^\gamma,\ldots,x_{k_\gamma}^\gamma)\in P(\mathbb{R}\setminus f_1^{-1}[\mathbb{R}])$. We can inductively (with respect to i) define $f_1(x_i^\gamma),\ldots,f_n(x_i^\gamma)$ such that $(a_1^\beta f_1+\cdots+a_n^\beta f_n)(x_i^\gamma)=y^\gamma$ for $i=1,\ldots,k_\gamma$. Indeed, choose

$$(f_1(x_i^\gamma),\ldots,f_n(x_i^\gamma))\in\{(y_1,\ldots,y_n)\in\mathbb{R}^n\colon a_1^\beta y_1+\cdots+a_n^\beta y_n=y^\gamma\}$$

such that

$$(f_1(x_i^\gamma),\ldots,f_n(x_i^\gamma))\notin\bigcup_{(x'_1,\ldots,x'_n)\in P(f_1^{-1}[\mathbb{R}])}(f_1(x'_1),\ldots,f_n(x'_1))+E_{(x'_1,\ldots,x'_n)},$$

where $E_{(x'_1,\dots,x'_n)} = \operatorname{span}\{(f_1(x'_i)-f_1(x'_{i+1}),\dots,f_n(x'_i)-f_n(x'_{i+1}))\colon i\le n-1\}$. Note that the above choice is possible since for each $(x'_1,\dots,x'_n)\in P(f_1^{-1}[\mathbb{R}])$, $\{(y_1,\dots,y_n)\in\mathbb{R}^n\colon a_1^\beta y_1+\cdots+a_n^\beta y_n = y^\gamma\}$ and $(f_1(x'_1),\dots,f_n(x'_1)) + E_{(x'_1,\dots,x'_n)}$ are two distinct affine hyperplanes (as otherwise we would have $|(a_1^\beta f_1+\cdots+a_n^\beta f_n)^{-1}(y^\gamma)| = n$).

This finishes the proof of the statement: for every $\beta\le\alpha$

$$|(a_1^\beta f_1+\cdots+a_n^\beta f_n)^{-1}(y)| = n \text{ for all } y\in(a_1^\beta f_1+\cdots+a_n^\beta f_n)[\{x_\gamma\colon\gamma<\alpha\}].$$

Hence the condition (c) holds for α and the inductive step of the definition of $f_1,\dots,f_n$ is completed. It follows from the construction that the conditions $(a)-(c)$ are preserved. The condition (a) assures that the functions $f_1,\dots,f_n$ are defined on $\mathbb{R}$ and the condition (c) assures that any nontrivial linear combination of $f_1,\dots,f_n$ is an n-to-one function. Hence the proof of $\mathcal{L}(\mathrm{F}_n) = n+1$ is completed.

To prove $\mathcal{L}_{\mathbb{Q}}(\mathrm{F}_n) = \mathfrak{c}^+$ first observe that in any family of functions of cardinality $>\mathfrak{c}$ there are two functions equal on a set of size $n+1$ (this follows from the fact that there are only $\mathfrak{c}$-many functions from a set of size $n+1$ into $\mathbb{R}$). Their difference is not in F_n. Hence $\mathcal{L}_{\mathbb{Q}}(\mathrm{F}_n)\le\mathfrak{c}^+$. To see that $\mathcal{L}_{\mathbb{Q}}(\mathrm{F}_n)\ge\mathfrak{c}^+$ consider a partition $\{A_\xi\colon\xi<\mathfrak{c}\}$ of $\mathbb{R}$ into subsets of size n and a partition $\{H_\alpha\colon|H_\alpha|=\mathfrak{c},\alpha<\mathfrak{c}\}$ of a Hamel basis. Define $f_\alpha\in\mathrm{F}_n$ such that $f_\alpha(\mathbb{R})\subseteq H_\alpha$ and $f_\alpha|A_\xi$ is constant for each $\xi<\mathfrak{c}$. It can be seen that f_α are linearly independent over $\mathbb{Q}$ and $\operatorname{span}_{\mathbb{Q}}\{f_\alpha\colon\alpha<\mathfrak{c}\}\subseteq\mathrm{F}_n\cup\{0\}$.

(ii) First, to see that $\mathcal{L}_{\mathbb{Q}}(\mathrm{F}_{<\omega})\le\mathfrak{c}^+$, observe that, similarly like above (at the end of proof of (i)), in any family of functions of cardinality $>\mathfrak{c}$ there are two functions equal on a set of size ω. Their difference is not in $\mathrm{F}_{<\omega}$.

To prove that $\mathcal{L}(\mathrm{F}_{<\omega})\ge\mathfrak{c}^+$ we will construct a family $\{f_\xi\colon\xi<\mathfrak{c}\}$ of functions such that for all $n\ge1$, $(a_1,a_2,\dots,a_n)\in\mathbb{R}^n\setminus\{(0,\dots,0)\}$, $\xi_1<\dots\xi_n<\mathfrak{c}$, and $y\in\mathbb{R}$ we have $|(a_1f_{\xi_1}+\cdots+a_nf_{\xi_n})^{-1}(\{y\})|\le n$. Fix $\alpha<\mathfrak{c}$ and assume that for every $\xi<\alpha$ the function f_ξ is defined on $\{x_\beta\colon\beta<\alpha\}$ and for all $n\ge1$, $(a_1,a_2,\dots,a_n)\in\mathbb{R}^n\setminus\{(0,\dots,0)\}$, $\xi_1<\dots\xi_n<\alpha$, and $y\in\mathbb{R}$ we have $|(a_1f_{\xi_1}+\cdots+a_nf_{\xi_n})^{-1}(\{y\})|\le n$. We will now define $f_\xi(x_\alpha)$ for every $\xi<\alpha$. Fix an $n\ge1$, $\xi_1<\cdots<\xi_n<\alpha$, $\beta_1<\cdots<\beta_n<\alpha$. For $n=1$ define $E_{\xi_1,\beta_1}=\{0\}$ and for $n\ge2$ set

$$E_{\xi_1,\dots\xi_n,\beta_1,\dots\beta_n} = \operatorname{span}\{(f_{\xi_1}(x_{\beta_i})-f_{\xi_1}(x_{\beta_{i+1}}),\dots,f_{\xi_n}(x_{\beta_i})-f_{\xi_n}(x_{\beta_{i+1}}))\colon i<n\}.$$

Note that $\dim(E_{\xi_1,\dots\xi_n,\beta_1,\dots\beta_n}) = n-1$. If $\dim(E_{\xi_1,\dots\xi_n,\beta_1,\dots\beta_n}) < n-1$ then we would get a contradiction with the inductive assumption as

$$(a_1f_{\xi_1}+\cdots+a_{n-1}f_{\xi_{n-1}})(x_{\beta_i}) = (a_1f_{\xi_1}+\cdots+a_{n-1}f_{\xi_{n-1}})(x_{\beta_{i+1}})$$

for $i \leq n-1$ and some $(a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{(0, \dots, 0)\}$. Indeed, the latter is equivalent to $\det([(f_{\xi_j}(x_{\beta_i}) - f_{\xi_j}(x_{\beta_{i+1}}))]_{i \leq n-1, j \leq n-1}) = 0$ which obviously is equivalent to $\dim(E_{\xi_1, \dots \xi_n, \beta_1, \dots \beta_n}) < n-1$. Define

$$\mathcal{E}_n = \{E_{\xi_1, \dots \xi_n, \beta_1, \dots \beta_n} \colon \xi_1 < \cdots < \xi_n < \alpha, \beta_1 < \cdots < \beta_n < \alpha\} \text{ and}$$

$$v_E = (f_{\xi_1}(x_{\beta_1}), \dots, f_{\xi_n}(x_{\beta_1}))$$

for $E = E_{\xi_1, \dots \xi_n, \beta_1, \dots \beta_n}$. Next apply Lemma 2.4 ($\zeta = \alpha$) to obtain $(y_0, y_1, \dots) \in (\mathbb{R} \setminus \{0\})^\alpha$ such that for every $n \geq 1$ and all $\xi_1 < \cdots < \xi_n < \alpha$ we have $(y_{\xi_1}, \dots, y_{\xi_n}) \notin \bigcup_{E \in \mathcal{E}_n} v_E + E$. Define $f_\xi(x_\beta) = y_\xi$.

As the next step we will define $f_\alpha(x_\beta)$ for every $\beta \leq \alpha$. Fix $\gamma \leq \alpha$ and assume that $f_\alpha(x_\beta)$ has been defined for every $\beta < \gamma$ in such a way that for all $n \geq 1$, $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n \setminus \{(0, \dots, 0)\}$, $\xi_1 < \dots \xi_{n-1} < \alpha$, and $y \in \mathbb{R}$ we have $|(a_1 f_{\xi_1} + \cdots + a_n f_\alpha)^{-1}(\{y\})| \leq n$. Pick $\beta_1 < \cdots < \beta_n < \gamma$ and notice that

$$\dim(\operatorname{span}\{(f_{\xi_1}(x_{\beta_i}) - f_{\xi_1}(x_{\beta_{i+1}}), \dots, f_\alpha(x_{\beta_i}) - f_\alpha(x_{\beta_{i+1}}))\colon i \leq n-1\}) = n-1.$$

This implies that there exists exactly one $y = y_{\xi_1, \dots, \xi_{n-1}, \alpha, \beta_1, \dots, \beta_n}$ such that $(f_{\xi_1}(x_{\beta_n}) - f_{\xi_1}(x_\gamma), \dots, f_\alpha(x_{\beta_n}) - y)$ is in

$$\operatorname{span}\{(f_{\xi_1}(x_{\beta_i}) - f_{\xi_1}(x_{\beta_{i+1}}), \dots, f_\alpha(x_{\beta_i}) - f_\alpha(x_{\beta_{i+1}}))\colon i \leq n-1\}.$$

Choose

$$f_\alpha(x_\gamma) \in \mathbb{R} \setminus \{y_{\xi_1, \dots, \xi_{n-1}, \alpha, \beta_1, \dots, \beta_n} \colon \xi_1 < \dots \xi_{n-1} < \alpha, \beta_1 < \cdots < \beta_n < \gamma\}.$$

This completes the step α of the definition of the family of functions $\{f_\xi \colon \xi < \mathfrak{c}\}$. It follows from the construction that the functions satisfy the desired property, namely: for all $n \geq 1$, $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n \setminus \{(0, \dots, 0)\}$, $\xi_1 < \dots \xi_n < \mathfrak{c}$, and $y \in \mathbb{R}$ we have $|(a_1 f_{\xi_1} + \cdots + a_n f_{\xi_n})^{-1}(\{y\})| \leq n$. Hence $\operatorname{span}\{f_\xi \colon \xi < \mathfrak{c}\} \subseteq F_{<\omega} \cup \{0\}$. $\square$

Corollary 2.5. *$\mathcal{L}(\mathrm{F}_{<n}) = n$ and $\mathcal{L}_{\mathbb{Q}}(\mathrm{F}_{<n}) = \mathfrak{c}^+$ for $n \geq 2$.*

Proof. The inequality $\mathcal{L}(\mathrm{F}_{<n}) \geq n$ is implied by the fact that $\mathrm{F}_{(n-1)} \subseteq \mathrm{F}_{<n}$ and Theorem 2.3 (i). The opposite inequality $\mathcal{L}(\mathrm{F}_{<n}) \geq n$ follows from the proof of the inequality $\mathcal{L}(\mathrm{F}_n) \leq n+1$ in part (i) of Theorem 2.3 (page 8).

The equality $\mathcal{L}_{\mathbb{Q}}(\mathrm{F}_{<n}) = \mathfrak{c}^+$ for $n \geq 2$ follows from Theorem 2.3 and the following observation $\mathrm{F}_{(n-1)} \subseteq \mathrm{F}_{<n} \subseteq \mathrm{F}_{<\omega}$. $\square$

Acknowledgments

The author wishes to thank the referee, whose thorough analysis and insightful remarks improved the article.

References

[1] R.M. Aron, V.I. Gurariy, J.B. Seoane-Sepúlveda, *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc. 133(3) (2005), 795–803.

[2] A. Bartoszewicz, S. Głąb, *Additivity and lineability in vector spaces*, Linear Algebra Appl. 439(7) (2013), 2123–2130.

[3] A. Bartoszewicz, S. Głąb, D. Pellegrino, J. B. Seoane-Sepúlveda, *Algebrability, non-linear properties, and special functions*, Proc. Amer. Math. Soc. 141(10) (2013), 3391–3402.

[4] L. Bernal-González, D. Pellegrino, J. B. Seoane-Sepúlveda, *Linear subsets of nonlinear sets in topological vector spaces*, Bull. Amer. Math. Soc. (N.S.) 51(1) (2014), 71–130.

[5] K.C. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Student Texts 39, Cambridge University Press.

[6] K.C. Ciesielski, J.L. Gámez-Merino, D. Pellegrino, and J.B. Seoane-Sepúlveda, *Lineability, spaceability, and additivity cardinals for Darboux-like functions*, Linear Algebra Appl. 440 (2014), 307-317

[7] J.L. Gámez-Merino, *Large algebraic structures inside the set of surjective functions*, Bull. Belg. Math. Soc. Simon Stevin 18(2) (2011), 297–300.

[8] J.L. Gámez-Merino, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, *Lineability and additivity in* $\mathbb{R}^{\mathbb{R}}$, J. Math. Anal. Appl. 369(1) (2010), 265–272.

[9] J.L. Gámez-Merino, G.A. Muñoz-Fernández, V.M. Sanchez, J.B. Seoane-Sepúlveda, *Sierpiński-Zygmund functions and other problems on lineability*, Proc. Amer. Math. Soc. 138 (11) (2010), 3863–3876.

[10] J. L. Gámez-Merino, J. B. Seoane-Sepúlveda, *An undecidable case of lineability in* $\mathbb{R}^{\mathbb{R}}$, J. Math. Anal. Appl. 401(2) (2013), 959–962.

[11] F.J. Garcia-Pacheco, M. Martin, J.B Seoane-Sepúlveda, *Lineability, spaceability and algebrability of certain subsets of function spaces*, Taiwanese J. Math. 13(4) (2009), 1257–1269.

[12] F.J. Garcia-Pacheco, N. Palmberg, J.B. Seoane-Sepúlveda, *Lineability and algebrability of pathological phenomena in analysis*, J. Math. Anal. Appl. 326 (2007) 929–939.

[13] V.I. Gurariy, L.Quarta, *On lineability of sets of continuous functions*, J. Math. Anal. Appl. 294 (2004), 62–72.

[14] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Birkhäuser Verlag AG, 2009.

[15] T. Natkaniec, *New cardinal invariants in real analysis*, Bull. Pol. Acad. Sci. Math. 44(2) (1996), 251–256.

[16] K. Płotka, *On functions whose graph is a Hamel basis*, Proc. Amer. Math. Soc. 131 (2003), 1031–1041.

[17] K. Płotka, *On functions whose graph is a Hamel basis II*, Canad. Math. Bull. 52(2) (2009), 295–302.

[18] K. Płotka, I. Recław, *Finitely continuous Hamel functions*, Real Anal. Exchange 30(2) (2004), 867–870.