ALGEBRAIC STRUCTURES WITHIN SUBSETS OF HAMEL AND SIERPIŃSKI-ZYGMUND FUNCTIONS

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Abstract. We prove the existence of an additive semigroup of cardinality $2^{\mathfrak{c}}$ contained in the intersection of the classes of Hamel functions (HF) and Sierpiński-Zygmund functions (SZ). In addition, we show that under certain set-theoretic assumptions the lineability of the class of Sierpiński-Zygmund functions (SZ) is equal to the lineability of the class of almost continuous Sierpiński-Zygmund functions (AC $\cap$ SZ).

1. Introduction

The symbols $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of positive integers, rational and real numbers, respectively. The cardinality of a set X is denoted by the symbol $|X|$. In particular, $|\mathbb{N}|$ is denoted by ω and $|\mathbb{R}|$ is denoted by $\mathfrak{c}$. We consider only real-valued functions. No distinction is made between a function and its graph. For any two partial real functions f, g we write $f+g$, $f-g$ for the sum and difference functions defined on $\text{dom}(f) \cap \text{dom}(g)$. We write $f|A$ for the restriction of f to the set $A \subseteq \mathbb{R}$. For any subset Y of a vector space V over the field E, any $v \in V$, and any $e \in E$ we define $v + Y = \{v + y : y \in Y\}$ and $eY = \{ey : y \in Y\}$.

Recently, there have been lots of attention devoted to finding "large" structures (e.g., vector spaces, algebras) contained in various families of real functions (see [1, 3–6, 8–10, 12, 16, 18]). In this article we also consider “less restrictive” structures like groups and even semigroups. In case of many classes of functions the problem is trivially solved by using already known results about vector spaces contained in those classes (as these vector spaces have maximal possible dimensions). However, in certain situations looking for the "largest" group or semigroup may be of interest.

We will recall here some of the most recent definitions related to the theory of lineability (see [3, 5, 6]). Let V be a vector space over the field E, $\mathcal{F} \subseteq V$, and κ be a cardinal number.

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We say that $\mathcal{F}$ is star-like (with respect to E) if $e\mathcal{F} \subseteq \mathcal{F}$ for all $e \in E \setminus \{0\}$. In addition, $\mathcal{F}$ is defined to be *κ-lineable* (over E) if $\mathcal{F} \cup \{0\}$ contains a subspace of V of dimension κ. The *(coefficient of) lineability of the subset $\mathcal{F}$ over the field E* is denoted by $\mathcal{L}_E(\mathcal{F})$ and defined as follows

$$\mathcal{L}_E(\mathcal{F}) = \min\{\kappa \colon \mathcal{F} \text{ is not } \kappa\text{-lineable over } E\}.$$

In the case $E = \mathbb{R}$ we simply write $\mathcal{L}(\mathcal{F})$.

Proposition 1.1. *Let V be a vector space over the field E_2 and E_1 be a subfield of E_2. If $\mathcal{F} \subseteq V$ is star-like with respect to E_2, then the following holds.*

(1)
$$\mathcal{L}_{E_1}(\mathcal{F}) \geq \begin{cases} ((\mathcal{L}_{E_2}(\mathcal{F}) - 1) \cdot \dim_{E_1}(E_2))^+ & \text{, if } \mathcal{L}_{E_2}(\mathcal{F}) < \omega \\ \mathcal{L}_{E_2}(\mathcal{F}) \cdot (\dim_{E_1}(E_2))^+ & \text{, otherwise.} \end{cases}$$

(2) *If E_1 is the smallest subfield of E_2 and $\mathcal{G}$ is an additive group contained in $\mathcal{F} \cup \{0\}$, then $E_1\mathcal{G} = \bigcup_{e \in E_1} e\mathcal{G}$ is a vector subspace of V over E_1 contained in $\mathcal{F} \cup \{0\}$.*

Proof. (1) Choose any $\kappa < \mathcal{L}_{E_2}(\mathcal{F})$ and let $W \subseteq V$ be a subspace over E_2 contained in $\mathcal{F} \cup \{0\}$ such that $\dim_{E_2}(W) = \kappa$. Obviously, W is also a subspace when considered over E_1 and it can be verified that

$$\dim_{E_1}(W) = \kappa \cdot \dim_{E_1}(E_2).$$

Indeed, if $\{f_\xi \colon \xi < \kappa\}$ is a basis of W over E_2, then $\{q_\lambda f_\xi \colon \xi < \kappa, \lambda < \dim_{E_1}(E_2)\}$ is a basis of W over E_1, where $\{q_\lambda \colon \lambda < \dim_{E_1}(E_2)\}$ is a basis of E_2 over E_1.

Now, if $\mathcal{L}_{E_2}(\mathcal{F}) < \omega$ then the largest possible κ is $\mathcal{L}_{E_2}(\mathcal{F}) - 1$ and in this case $\dim_{E_1}(W) = (\mathcal{L}_{E_2}(\mathcal{F}) - 1) \cdot \dim_{E_1}(E_2)$ and consequently $\mathcal{L}_{E_1}(\mathcal{F}) \geq ((\mathcal{L}_{E_2}(\mathcal{F}) - 1) \cdot \dim_{E_1}(E_2))^+$.

If $\mathcal{L}_{E_2}(\mathcal{F}) \geq \omega$, then $\mathcal{L}_{E_1}(\mathcal{F}) \geq \max\{\mathcal{L}_{E_2}(\mathcal{F}), (\dim_{E_1}(E_2))^+\} = \mathcal{L}_{E_2}(\mathcal{F}) \cdot (\dim_{E_1}(E_2))^+$.

(2) Since $\mathcal{G} \subseteq \mathcal{F} \cup \{0\}$ and $\mathcal{F}$ is star-like obviously $E_1\mathcal{G} \subseteq \mathcal{F} \cup \{0\}$. Additionally, observe that

$$E_1 = \left\{ \pm\frac{e_n}{e_k} \colon k, n \in \mathbb{Z}_+ \text{ and } e_k \neq 0, \text{ where } e_i \text{ is the sum of } i \text{ 1's} \right\} \cup \{0\}.$$

Therefore, for all $g_1, g_2 \in \{\mathcal{G}\}$ and $q_1, q_2 \in E_1 \setminus \{0\}$ we have

$$q_1 g_1 + q_2 g_2 = \pm\frac{e_{n_1}}{e_{k_1}} g_1 \pm \frac{e_{n_2}}{e_{k_2}} g_2 = \frac{1}{e_{k_1} e_{k_2}} (\pm e_{n_1} e_{k_2} g_1 \pm e_{n_2} e_{k_1} g_2) \in E_1\mathcal{G}.$$

Hence, $E_1\mathcal{G}$ is a vector subspace of V over E_1 contained in $\mathcal{F} \cup \{0\}$. □

Observe that in general, the weak inequality in part (1) cannot be replaced by equality neither strict inequality. Indeed, if $\mathcal{F}$ is a vector space over E_2, then there is equality in part (1). On the other hand if we pick $V = \mathbb{R}^{\mathbb{R}}$, $E_1 = \mathbb{Q}$, $E_2 = \mathbb{R}$, B to be a basis of V over E_2, and define $\mathcal{F} = \operatorname{span}_{\mathbb{Q}}(B) \cup \bigcup_{e\in\mathbb{R}} eB$ then we have $\mathcal{L}_{E_1}(\mathcal{F}) = 2^{\mathfrak{c}}$ and $\mathcal{L}_{E_2}(\mathcal{F}) = 2$. Hence in that case there is strict inequality $>$ in part (1).

As a consequence of the above proposition, let us note here that if $E_1 = \mathbb{Q}$ and $E_2 = \mathbb{R}$ then for any star-like $\mathcal{F}$ we have that $\mathcal{L}_{\mathbb{Q}}(\mathcal{F}) \geq (\dim_{\mathbb{Q}}(\mathbb{R}))^+ = \mathfrak{c}^+$. Additionally, every additive group contained in $\mathcal{F} \cup \{0\}$ has cardinality less than $\mathcal{L}_{\mathbb{Q}}(\mathcal{F})$.

In this article we consider the following classes of functions. A function $f\colon \mathbb{R} \to \mathbb{R}$ is:

- an *extendability* function provided there exists a connectivity function $F\colon \mathbb{R}\times[0,1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in \mathbb{R}$ ($f \in \mathrm{Ext}$);
- *almost continuous* (in sense of Stallings) if each open subset of $\mathbb{R}^2$ containing the graph of f contains also the graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$ ($f \in \mathrm{AC}$);
- *Hamel* function if the graph of f is a Hamel basis for $\mathbb{R}^2$ ($f \in \mathrm{HF}$);
- *Sierpiński-Zygmund* if for every set $Y \subseteq \mathbb{R}$ of cardinality continuum $\mathfrak{c}$, $f|Y$ is discontinuous ($f \in \mathrm{SZ}$).

Recall here that the class of all continuous functions is contained in Ext, $\mathrm{Ext} \subseteq \mathrm{AC}$, $\mathrm{Ext}\cap\mathrm{SZ} = \emptyset$, $\mathrm{AC}\cap\mathrm{SZ} \neq \emptyset$ under additional set-theoretical assumptions (e.g., CH, Martin's Axiom), $\mathrm{Ext} \cap \mathrm{HF} \neq \emptyset$, $\mathrm{AC} \cap \mathrm{HF} \neq \emptyset$, and $\mathrm{HF} \cap \mathrm{SZ} \neq \emptyset$ (see [17]). In addition, a function $f\colon \mathbb{R} \to \mathbb{R}$ is almost continuous if and only if it intersects every *blocking set*, i.e., a closed set $K \subseteq \mathbb{R}^2$ which meets every continuous function and is disjoint with at least one function from $\mathbb{R}$ to $\mathbb{R}$. The domain of every blocking set contains a non-degenerate connected set. (See [11].) For $f \in \mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ we say that a set $A \subseteq \mathbb{R}$ is f-negligible with respect to $\mathcal{F}$ if for every function g such that $f|(\mathbb{R} \setminus A) \equiv g|(\mathbb{R} \setminus A)$ we have that $g \in \mathcal{F}$.

It is known that $\mathcal{L}(\mathrm{SZ}) > \mathfrak{c}^+$ (see [9]) and that $2^{\mathfrak{c}}$-lineability of SZ is undecidable in ZFC (see [10]). In [10] the authors also proved (Theorem 2.2) that for any $\mathfrak{c} < \kappa \leq 2^{\mathfrak{c}}$, $\mathcal{L}(\mathrm{SZ}) > \kappa$ is equivalent to the existence of an additive group in $\mathrm{SZ} \cup \{0\}$ of cardinality κ. This immediately implies the following property.

Remark 1.2. $\mathcal{L}_{\mathbb{Q}}(\mathrm{SZ}) = \mathcal{L}(\mathrm{SZ})$.

In the case of Hamel functions we have the following: $\mathcal{L}(\mathrm{HF}) = 2$ and $\mathcal{L}_{\mathbb{Q}}(\mathrm{HF}) = \mathfrak{c}^+$ (see [18]).

2. Semigroup in $\mathrm{HF} \cap \mathrm{SZ}$ and lineability of $\mathrm{AC} \cap SZ$

We will be using the following two lemmas to prove the existence of "large" semigroups in $\mathrm{HF} \cap \mathrm{SZ}$, $\mathrm{Ext} \cap \mathrm{HF}$, and $\mathrm{AC} \cap \mathrm{HF} \cap \mathrm{SZ}$ (under the assumption of CH).

Lemma 2.1. [17, Lemma 7] *Let $V \subseteq \mathbb{R}^n$ be a Hamel basis and $v' \in V$. For each $v \in V$ fix $q_v \in \mathbb{Q}$ such that $q_{v'} \neq -1$. Then the set $V' = \{v + q_v v' \colon v \in V\}$ is also a Hamel basis.*

Lemma 2.2. *There exists a function $h \in \mathrm{HF} \cap \mathrm{Ext}$ and a set $X \subseteq \mathbb{R}$ of cardinality $\mathfrak{c}$ which is h-negligible with respect to* Ext. *Assuming CH, there exists a function $h \in \mathrm{AC} \cap \mathrm{HF} \cap \mathrm{SZ}$ and a set $X \subseteq \mathbb{R}$ of cardinality $\mathfrak{c}$ which is h-negligible with respect to* AC.

Proof. Let $F \subseteq \mathbb{R}$ be a linearly independent $\mathfrak{c}$-dense F_σ set (see [14, Theorem 11.7.2]). Then there exists a function $f \in \mathrm{Ext}$ such that $\mathbb{R} \setminus F$ is f-negilgible (see [7]). Using [17, Fact 6] we obtain the existence of a function $h \in \mathrm{HF}$ such that $h|(\mathbb{R} \setminus F) \equiv f|(\mathbb{R} \setminus F)$. Obviously, $h \in \mathrm{Ext}$ and $X = \mathbb{R} \setminus F$ is h-negligible with respect to Ext.

In the proof of Theorem 2 in [17] (page 123) a function h is constructed which belongs to $\mathrm{AC} \cap \mathrm{HF} \cap \mathrm{SZ}$ (under CH). One can easily see that this function h has a dense graph. It is known that for an almost continuous function f with a dense graph, every nowhere dense set is f-negligible with respect to AC (see [13]). □

Theorem 2.3. *Both* $\mathrm{HF} \cap \mathrm{Ext}$ *and* $\mathrm{HF} \cap \mathrm{SZ}$ *contain an additive semigroup of size $2^{\mathfrak{c}}$. In addition, assuming CH, the same holds for* $\mathrm{AC} \cap \mathrm{HF} \cap \mathrm{SZ}$.

Proof. We will prove the statement for the family $\mathrm{AC} \cap \mathrm{HF} \cap \mathrm{SZ}$. By the previous lemma, under the assumption of CH there exists a function $h \in \mathrm{AC} \cap \mathrm{HF} \cap \mathrm{SZ}$ and a set $X \subseteq \mathbb{R}$ of cardinality $\mathfrak{c}$ which is h-negligible with respect to AC. Define $\mathcal{H} = \{qh + h(0)g \colon q \in \mathbb{Q}_+,\ g \in \mathbb{Q}_+(X)\}$ where $\mathbb{Q}_+$ is the set of positive rationals and $\mathbb{Q}_+(X) = \{f \in \mathbb{R}^{\mathbb{R}} \colon f|(\mathbb{R} \setminus X) \equiv 0$ and $f(x) \in \mathbb{Q}_+$ for $x \in X\}$. Since $h(0) \neq 0$ (for every $f \in \mathrm{HF}$ $f(0) \neq 0$) we conclude that $|\mathcal{H}| = 2^{\mathfrak{c}}$. Next observe that $\mathcal{H}$ is closed under addition as both $\mathbb{Q}_+$ and $\mathbb{Q}_+(X)$ are closed under addition.

Finally we will justify that $\mathcal{H} \subseteq \mathrm{AC} \cap \mathrm{HF} \cap \mathrm{SZ}$. Obviously $\mathcal{H} \subseteq \mathrm{AC}$ as AC is star-like and X is h-negligible with respect to AC. To see $\mathcal{H} \subseteq \mathrm{HF}$ recall that HF is star-like and then use Lemma 2.1 with $V = h$, $v' = (0, h(0))$, and $q_v = \frac{g(x)}{q}$ for $v = (x, h(x))$, $q \in \mathbb{Q}_+$, $g \in \mathbb{Q}_+(X)$ to conclude that $h + h(0)\frac{g}{q}$ is a Hamel function ($h + h(0)\frac{g}{q}$ is V' from Lemma 2.1). Consequently, $qh + h(0)g = q(h + h(0)\frac{g}{q}) \in \mathrm{HF}$.

To see $\mathcal{H} \subseteq \mathrm{SZ}$ recall that SZ is star-like and observe that $h(0)g$ is a countably continuous function (e.g., union of countably many partial continuous functions) for all $q \in \mathbb{Q}_+$ and $g \in \mathbb{Q}_+(X)$. This implies $qh + h(0)g \in \mathrm{SZ}$.

The existence of semigroups of cardinality $2^{\mathfrak{c}}$ in $\mathrm{HF} \cap \mathrm{Ext}$ and $\mathrm{HF} \cap \mathrm{SZ}$ can be justified in a very similar way (in the case of $\mathrm{HF} \cap \mathrm{SZ}$ use $X = \mathbb{R}$). □

Theorem 2.4. *Assume CH. Then $\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) > \mathfrak{c}^+$.*

Proof. Let $\mathcal{F} = \{f_\gamma : \gamma < \mathfrak{c}\} \subseteq (\mathrm{AC} \cap \mathrm{SZ}) \cup \{0\}$ be a vector space of dimension $\leq \mathfrak{c}$. We will show that there exists an $h \in \mathrm{AC} \cap \mathrm{SZ} \setminus \mathcal{F}$ such that $h + \mathcal{F} \subseteq \mathrm{AC} \cap \mathrm{SZ}$. Since $\mathrm{AC} \cap \mathrm{SZ}$ is star-like the latter will imply that $\{ah\colon \in \mathbb{R}\} + \mathcal{F}$ is a vector space in $(\mathrm{AC} \cap \mathrm{SZ}) \cup \{0\}$ such that $\mathcal{F} \subsetneq \{ah\colon \in \mathbb{R}\} + \mathcal{F}$. Next using Zorn's lemma we will be able to conclude that $(\mathrm{AC} \cap \mathrm{SZ}) \cup \{0\}$ contains a vector space of dimension $\mathfrak{c}^+$.

Let $\mathcal{G} = \{g_\alpha : \alpha < \mathfrak{c}\}$ be the set of all continuous functions defined on G_δ subsets of $\mathbb{R} = \{x_\alpha : \alpha < \mathfrak{c}\}$. For every $\alpha < \mathfrak{c}$ define U_α to be the maximal open set such that $\mathrm{dom}(g_\alpha \setminus \bigcup_{\xi<\alpha} g_\xi)$ is residual in U_α. We will construct by induction a sequence of partial functions h_α $(\alpha < \mathfrak{c})$ such that:

(i) $h_\xi \subseteq h_\alpha$ for $\xi < \alpha$;

(ii) $|\mathrm{dom}(h_\alpha)| \leq \omega$ and $x_\alpha \in \mathrm{dom}(h_\alpha)$;

(iii) $(g_\zeta \cap (f_\gamma + h_\alpha)) \subseteq (f_\gamma + h_\xi)$ for $\zeta, \gamma \leq \xi < \alpha$;

(iv) $f_\gamma + h_\alpha$ is dense subset of $(g_\zeta \setminus \bigcup_{\xi<\zeta} g_\xi)|U_\zeta$ for $\zeta, \gamma \leq \alpha$.

We start the construction of the sequence h_α $(\alpha < \mathfrak{c})$ by defining $h_0(x_0)$ arbitrarily. Next choose a countable dense subset $D_0 \subseteq (\mathrm{dom}(g_0) \cap U_0) \setminus \{x_0\}$ and put $(f_0 + h_0)|D_0 \equiv g_0|D_0$ (or equivalently $h_0|D_0 \equiv (g_0 - f_0)|D_0$). It is easy to see that h_0 satisfies all the conditions (i)-(iv).

Now fix $\alpha < \mathfrak{c}$ and assume that the sequence h_β has been defined for all $\beta < \alpha$ satisfying the conditions (i)-(iv). Put $h_\alpha = \bigcup_{\beta<\alpha} h_\beta$. If $x_\alpha \notin \mathrm{dom}(h_\alpha)$, then choose

$$h_\alpha(x_\alpha) \in \mathbb{R} \setminus \bigcup_{\gamma,\beta<\alpha} \{g_\beta(x_\alpha) - f_\gamma(x_\alpha)\}).$$

Next notice that since the conditions (i)-(iv) are satisfied for all $\beta < \alpha$ to have that

$$f_\gamma + h_\alpha \text{ is dense in } (g_\zeta \setminus \bigcup_{\xi<\zeta} g_\xi)|U_\zeta \text{ for } \zeta, \gamma \leq \alpha$$

it suffices to assure the above condition for $\zeta = \alpha$ or $\gamma = \alpha$. Choose a collection of pairwise disjoint countable sets $D_{\gamma,\xi}$ (($\gamma < \alpha$ and $\xi = \alpha$) or ($\gamma = \alpha$ and $\xi \leq \alpha$)) contained in

$$\mathbb{R} \setminus \left(\mathrm{dom}(h_\alpha) \cup \bigcup_{\xi_1,\xi_2,\gamma_1,\gamma_2 \leq \alpha, \gamma_1 \neq \gamma_2} \mathrm{dom}((g_{\xi_1} - g_{\xi_2}) \cap (f_{\gamma_1} - f_{\gamma_2})) \right),$$

such that $D_{\gamma,\alpha}$ is dense subset of $\mathrm{dom}(g_\alpha \setminus \bigcup_{\beta<\alpha} g_\beta) \cap U_\alpha$ ($\gamma < \alpha$) and $D_{\alpha,\xi}$ is dense subset of $\mathrm{dom}(g_\xi \setminus \bigcup_{\beta<\xi} g_\beta) \cap U_\xi$ ($\xi \leq \alpha$). Note here that the above choice is possible as $|\mathrm{dom}((g_{\xi_1} - g_{\xi_2}) \cap (f_{\gamma_1} - f_{\gamma_2}))| \leq \omega$ for $\xi_1, \xi_2, \gamma_1, \gamma_2 \leq \alpha, \gamma_1 \neq \gamma_2$ because $g_{\xi_1} - g_{\xi_2}$ is a continuous function, $f_{\gamma_1} - f_{\gamma_2} \in \mathrm{SZ}$, and we work under the assumption of CH. Now we define $f_\gamma + h_\alpha|D_{\gamma,\alpha} \equiv g_\alpha|D_{\gamma,\alpha}$ for $\gamma < \alpha$ and $f_\alpha + h_\alpha|D_{\alpha,\xi} \equiv g_\xi|D_{\alpha,\xi}$ for $\xi \leq \alpha$. This finishes the construction of h_α. It is clear that h_α satisfies the conditions (i), (ii), and (iv).

To see that the condition (iii) is also satisfied let us pick $\xi < \alpha$ and $\zeta, \gamma \leq \xi$. By the inductive assumption we obtain that $(g_\zeta \cap (f_\gamma + \bigcup_{\beta<\alpha} h_\beta)) \subseteq (f_\gamma + h_\xi)$. Therefore to conclude that $(g_\zeta \cap (f_\gamma + h_\alpha)) \subseteq (f_\gamma + h_\xi)$ we need to justify that $g_\zeta \cap (f_\gamma + h_\alpha|\{x_\alpha\}) = \emptyset$, $g_\zeta \cap (f_\gamma + h_\alpha|D_{\gamma_1,\alpha}) = \emptyset$ ($\gamma_1 < \alpha$), and $g_\zeta \cap (f_\gamma + h_\alpha|D_{\alpha,\xi}) = \emptyset$ ($\xi \leq \alpha$). The equality $g_\zeta \cap (f_\gamma + h_\alpha|\{x_\alpha\}) = \emptyset$ easily follows from the definition of $h_\alpha(x_\alpha)$. To see $g_\zeta \cap (f_\gamma + h_\alpha|D_{\gamma_1,\alpha}) = \emptyset$ ($\gamma_1 < \alpha$) note that $g_\zeta \cap (f_\gamma + h_\alpha|D_{\gamma_1,\alpha}) = g_\zeta \cap (f_\gamma - f_{\gamma_1} + g_\alpha)|D_{\gamma_1,\alpha}$. If $\gamma = \gamma_1$, then $g_\zeta \cap (f_\gamma + h_\alpha|D_{\gamma_1,\alpha}) = g_\zeta \cap g_\alpha|D_{\gamma_1,\alpha} = \emptyset$ as $D_{\gamma_1,\alpha} \subseteq \mathrm{dom}(g_\alpha \setminus \bigcup_{\beta<\alpha} g_\beta)$. If $\gamma \neq \gamma_1$, then $g_\zeta \cap (f_\gamma + h_\alpha|D_{\gamma_1,\alpha}) = (g_\zeta - g_\alpha) \cap (f_\gamma - f_{\gamma_1})|D_{\gamma_1,\alpha} = \emptyset$ as

$$D_{\gamma_1,\alpha} \cap \bigcup_{\xi_1,\xi_2,\gamma_1,\gamma_2 \leq \alpha, \gamma_1 \neq \gamma_2} \mathrm{dom}((g_{\xi_1} - g_{\xi_2}) \cap (f_{\gamma_1} - f_{\gamma_2})) = \emptyset.$$

Very similarly we can justify that $g_\zeta \cap (f_\gamma + h_\alpha|D_{\alpha,\xi}) = \emptyset$ ($\xi \leq \alpha$). Hence the condition (iii) holds for h_α. This finishes the inductive definition of the sequence h_α ($\alpha < \mathfrak{c}$) satisfying the conditions (i)-(iv).

Define $h = \bigcup_{\alpha<\mathfrak{c}} h_\alpha$. Obviously $\mathrm{dom}(h) = \mathbb{R}$. The conditions (ii)-(iii) imply that $h + f_\gamma \in$ SZ for all $\gamma < \mathfrak{c}$ as any partial continuous function can be extended to a continuous function on a G_δ subset of $\mathbb{R}$ (see [15]) and $(g_\zeta \cap (f_\gamma + h)) \subseteq (f_\gamma + h_{\max(\zeta,\gamma)})$ for all $\zeta < \mathfrak{c}$.

Next we will argue that $h + f_\gamma$ is almost continuous for every $\gamma < \mathfrak{c}$. Let $B \subseteq \mathbb{R}^2$ be any blocking set. There exists a non-empty open interval $I \subseteq \mathrm{dom}(B)$ and a continuous function g such that $\mathrm{dom}(g)$ is G_δ dense subset of I and $g \subseteq B$. Let ζ_0 be the smallest ordinal number with the property that $g_{\zeta_0}|I \subseteq B$ and $\mathrm{dom}(g_{\zeta_0}) \cap I$ is residual in I for some non-empty open interval $I \subseteq \mathrm{dom}(B)$. Then $\mathrm{dom}(g_{\zeta_0} \setminus \bigcup_{\xi<\zeta_0} g_\xi)$ is also residual in I (since we assume CH). Therefore, $I \subseteq U_{\zeta_0}$ and since $f_\gamma + h$ is dense subset of $(g_{\zeta_0} \setminus \bigcup_{\xi<\zeta_0} g_\xi)|U_{\zeta_0}$ (condition (iv) for $\alpha = \max(\gamma, \zeta_0)$) we obtain that

$$\emptyset \neq (h + f_\gamma) \cap (g_{\zeta_0} \setminus \bigcup_{\xi<\zeta_0} g_\xi)|I \subseteq (h + f_\gamma) \cap g_{\zeta_0}|I \subseteq (h + f_\gamma) \cap B.$$

This implies that $h + f_\gamma \in$ AC. □

Let us mention here that assuming GHC the above theorem implies that AC $\cap$ SZ is $2^{\mathfrak{c}}$-lineable and consequently $\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) = \mathcal{L}(\mathrm{SZ})$. On the other hand, there is a model of ZFC (see [2]) in which AC $\cap$ SZ $= \emptyset$. These two observations imply the following.

Corollary 2.5.

(1) *It is consistent with ZFC that* $\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) = \mathcal{L}(\mathrm{SZ})$.

(2) *It is consistent with ZFC that* $\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) < \mathcal{L}(\mathrm{SZ})$.

It would be interesting to know if it is possible to have AC $\cap$ SZ $\neq \emptyset$ and $\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) < \mathcal{L}(\mathrm{SZ})$. We state that as an open problem.

Problem 2.6. Is it consistent with ZFC that AC $\cap$ SZ $\neq \emptyset$ and $\mathcal{L}(\mathrm{AC} \cap \mathrm{SZ}) < \mathcal{L}(\mathrm{SZ})$?

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