Sum of Sierpiński-Zygmund and Darboux Like Functions

Krzysztof Płotka*
Department of Mathematics, West Virginia University
Morgantown, WV 26506-6310, USA
kplotka@math.wvu.edu
and
Institute of Mathematics, Gdańsk University
Wita Stwosza 57, 80-952 Gdańsk, Poland

February 28, 2000

Abstract

For $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathbb{R}^{\mathbb{R}}$ we define $\mathrm{Add}(\mathcal{F}_1, \mathcal{F}_2)$ as the smallest cardinality of a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ for which there is no $g \in \mathcal{F}_1$ such that $g + F \subseteq \mathcal{F}_2$. The main goal of this note is to investigate the function Add in the case when one of the classes $\mathcal{F}_1$, $\mathcal{F}_2$ is the class SZ of *Sierpiński-Zygmund* functions. In particular, we show that *Martin's Axiom* (MA) implies $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) \geq \omega$ and $\mathrm{Add}(\mathrm{SZ}, \mathrm{AC}) = \mathrm{Add}(\mathrm{SZ}, \mathrm{D}) = \mathfrak{c}$, where AC and D denote the families of *almost continuous* and *Darboux* functions, respectively. As a corollary we obtain that the proposition: *every function from* $\mathbb{R}$ *into* $\mathbb{R}$ *can be represented as a sum of Sierpiński-Zygmund and almost continuous functions* is independent of ZFC axioms.

1 Introduction

The terminology is standard and follows [2]. The symbols $\mathbb{R}$ and $\mathbb{Q}$ stand for the sets of all real and all rational numbers, respectively. A basis of $\mathbb{R}$ as a linear space over $\mathbb{Q}$ is called *Hamel basis*. For $Y \subset \mathbb{R}$, the symbol $\mathrm{Lin}_{\mathbb{Q}}(Y)$ stands for the smallest linear subspace of $\mathbb{R}$ over $\mathbb{Q}$ that contains Y. The cardinality of a set X we denote by $|X|$. In particular, $|\mathbb{R}|$ is denoted by $\mathfrak{c}$. Given a cardinal κ, we let $\mathrm{cf}(\kappa)$ denote the cofinality of κ. We say that a cardinal κ is regular provided that $\mathrm{cf}(\kappa) = \kappa$.

$\mathcal{B}$ and $\mathcal{M}$ stand for the families of all Borel and all meager subsets of $\mathbb{R}$, respectively. We say that a set $B \subseteq \mathbb{R}$ is a *Bernstein* set if both B and $\mathbb{R} \setminus B$

*This paper was written under supervision of K. Ciesielski. The author wishes to thank him for many helpful conversations.

intersect every perfect set. For a cardinal number κ, a set $A \subseteq \mathbb{R}$ is called *κ-dense* if $|A \cap I| \geq \kappa$ for every non-trivial interval I. For any planar set P, we denote its x-projection by $\mathrm{dom}(P)$.

We consider only real-valued functions. No distinction is made between a function and its graph. For any two partial real functions f, g we write $f+g$, $f-g$ for the sum and difference functions defined on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. The class of all functions from a set X into a set Y is denoted by Y^X. We write $f|A$ for the restriction of $f \in Y^X$ to the set $A \subseteq X$. For $B \subseteq \mathbb{R}^n$ its characteristic function is denoted by χ_B. If $f, g \in Y^X$, we denote the set $\{x \in X : f(x) = g(x)\}$ by $[f = g]$. For any function $g \in \mathbb{R}^X$ and any family of functions $F \subseteq \mathbb{R}^X$ we define $g + F = \{g + f \colon f \in F\}$.

The cardinal function $\mathrm{A}(\mathcal{F})$, for $\mathcal{F} \subseteq \mathbb{R}^X$, is defined as the smallest cardinality of a family $F \subseteq \mathbb{R}^X$ for which there is no $g \in \mathbb{R}^X$ such that $g + F \subseteq \mathcal{F}$. It was investigated for many different classes of real functions, see e.g. [5], [6], [13]. In this paper we generalize the function A by imposing some restrictions on the function g. Thus for $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathbb{R}^X$ we define

$$\mathrm{Add}(\mathcal{F}_1, \mathcal{F}_2) = \min \{|F| \colon F \subseteq \mathbb{R}^X \ \& \ \neg\exists g \in \mathcal{F}_1 \ g + F \subseteq \mathcal{F}_2\} \cup \{(|\mathbb{R}^X|)^+\}.$$

Observe that $\mathrm{A}(\mathcal{F}) = \mathrm{Add}(\mathbb{R}^X, \mathcal{F})$ for any set X, so the function Add is indeed a generalization of the function A. Notice also the following properties of the Add function.

Proposition 1 *Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathbb{R}^X$ and $\mathcal{F} \subseteq \mathbb{R}^X$.*

(1) $\mathrm{Add}(\mathcal{F}_1, \mathcal{F}) \leq \mathrm{Add}(\mathcal{F}_2, \mathcal{F})$.

(2) $\mathrm{Add}(\mathcal{F}, \mathcal{F}_1) \leq \mathrm{Add}(\mathcal{F}, \mathcal{F}_2)$.

(3) $\mathrm{Add}(\mathcal{F}_1, \mathcal{F}_2) \geq 2$ *if and only if* $\mathbb{R}^X = \mathcal{F}_2 - \mathcal{F}_1$.

(4) *If* $\mathrm{Add}(\mathcal{F}_1, \mathcal{F}_2) \geq 2$ *then* $\mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$.

(5) $\mathrm{A}(\mathcal{F}) = \mathrm{Add}(\mathcal{F}, \mathcal{F}) + 1$. *In particular, if* $\mathrm{A}(\mathcal{F}) \geq \omega$ *then* $\mathrm{Add}(\mathcal{F}, \mathcal{F}) = \mathrm{A}(\mathcal{F})$.[1]

PROOF. The properties (1)-(4) are obvious. We will prove (5). It is clear that $\mathrm{Add}(\mathcal{F}, \mathcal{F}) \leq \mathrm{A}(\mathcal{F})$. On the other hand, observe that $\mathrm{A}(\mathcal{F}) \leq \mathrm{Add}(\mathcal{F}, \mathcal{F}) + 1$. To see the above let $F \subseteq \mathbb{R}^X$ be such that $|F| = \mathrm{Add}(\mathcal{F}, \mathcal{F})$ and

$$\neg\exists \ g \in \mathcal{F} \ \ g + F \subseteq \mathcal{F}.$$

Then we have

$$\neg\exists \ g \in \mathbb{R}^X \ \ g + (F \cup \{\mathbf{0}\}) \subseteq \mathcal{F},$$

where $\mathbf{0} \colon X \to \mathbb{R}$ is a function identically equal to zero.

[1] Very similar observation, in a little bit different context, was obtained independently by Francis Jordan [8, Proposition 1.3].

So the conclusion is obvious in the case $\mathrm{A}(\mathcal{F}) \geq \omega$. Therefore we will concentrate on the case $\mathrm{A}(\mathcal{F}) = k$ for some $k \in \omega$. Recall that the function A is bounded from the bottom by 1, thus $k \geq 1$. From the previous argument we imply that $\mathrm{Add}(\mathcal{F}, \mathcal{F}) \geq k-1$. So we only need to justify that $\mathrm{Add}(\mathcal{F}, \mathcal{F}) \leq k-1$.

Let $\{f_1, \ldots, f_k\}$ be a family witnessing $\mathrm{A}(\mathcal{F}) = k$. Then the set $\{f_1 - f_k, \ldots, f_{k-1} - f_k\}$ witnesses $\mathrm{Add}(\mathcal{F}, \mathcal{F}) \leq k-1$. Indeed, assume by contradiction, that we can find a function $f \in \mathcal{F}$ such that $(f_i - f_k) + f \in \mathcal{F}$ for every $i = 1, \ldots, k-1$. Then the function $f - f_k$ shifts the set $\{f_1, \ldots, f_k\}$ into $\mathcal{F}$. Contradiction. ■

Our main goal is to investigate the function Add in the case when one of the classes $\mathcal{F}_1$, $\mathcal{F}_2$ is the class of *Sierpiński-Zygmund* functions. Before we state the main result of the paper, let us recall the following definitions.

For $X \subseteq \mathbb{R}^n$ a function $f\colon X \to \mathbb{R}$ is:

- *additive* if $f(x+y) = f(x) + f(y)$ for all $x, y \in X$ such that $x + y \in X$;
- *almost continuous* (in sense of Stallings) if each open subset of $X \times \mathbb{R}$ containing the graph of f contains also graph of a continuous function from X to $\mathbb{R}$;
- *connectivity* if the graph of $f|Z$ is connected in $Z \times \mathbb{R}$ for any connected subset Z of X;
- *countably continuous* if it can be represented as a union of countably many continuous partial functions;
- *Darboux* if $f[K]$ is a connected subset of $\mathbb{R}$ (i.e., an interval) for every connected subset K of X;
- an *extendability* function provided there exists a connectivity function $F\colon X \times [0,1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in X$;
- *peripherally continuous* if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset W of U such that $x \in W$ and $f[\mathrm{bd}(W)] \subset V$;
- *Sierpiński-Zygmund* if for every set $Y \subseteq X$ of cardinality continuum $\mathfrak{c}$, $f|Y$ is discontinuous.

The classes of functions defined above are denoted by $\mathrm{AD}(X)$, $\mathrm{AC}(X)$, $\mathrm{Conn}(X)$, $\mathrm{CC}(X)$, $\mathrm{D}(X)$, $\mathrm{Ext}(X)$, $\mathrm{PC}(X)$, and $\mathrm{SZ}(X)$, respectively. The family of all continuous functions from X into $\mathbb{R}$ is denoted by $\mathrm{C}(X)$. We drop the index X in the case $X = \mathbb{R}$. To simplify notation, we introduce the symbols $\mathrm{SZ}_{\mathrm{part}}$ and $\mathrm{CC}_{\mathrm{part}}$ to denote $\bigcup_{X \subseteq \mathbb{R}} \mathrm{SZ}(X)$ and $\bigcup_{X \subseteq \mathbb{R}} \mathrm{CC}(X)$.

Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is almost continuous if and only if it intersects every *blocking set*, i.e., a closed set $K \subseteq \mathbb{R}^{n+1}$ which meets every continuous function from $\mathrm{C}(\mathbb{R}^n)$ and is disjoint with at least one function from $\mathbb{R}^{\mathbb{R}^n}$. The domain of every blocking set contains a non-degenerate connected set. (See [10].) It is also well-known that each continuous partial function can be extended to a continuous function defined on some G_δ set. (See [12].) Thus if $|[f = g]| < \mathfrak{c}$ for each continuous partial function g defined on some G_δ-set then f is Sierpiński-Zygmund. Recall also that each additive function $f \in \mathrm{AD}$ is linear over $\mathbb{Q}$, i.e., for all $p, q \in \mathbb{Q}$ and $x, y \in \mathbb{R}$ we have $f(px + qy) = pf(x) + qf(y)$.

The above classes are related in the following way (arrows $\longrightarrow$ indicate proper inclusions.) (See [3] or [7].)

$$\mathrm{C} \longrightarrow \mathrm{Ext} \longrightarrow \mathrm{AC} \longrightarrow \mathrm{Conn} \longrightarrow \mathrm{D} \longrightarrow \mathrm{PC}$$

For functions from $\mathbb{R}$ into $\mathbb{R}$.

$$\begin{tikzcd} & & & \mathrm{AC}(\mathbb{R}^n) \\ \mathrm{C}(\mathbb{R}^n) \arrow[r] & \mathrm{Ext}(\mathbb{R}^n) = \mathrm{Conn}(\mathbb{R}^n) = \mathrm{PC}(\mathbb{R}^n) \arrow[r] & \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) \arrow[ur] \arrow[dr] & \\ & & & \mathrm{D}(\mathbb{R}^n) \end{tikzcd}$$

For functions from $\mathbb{R}^n$ into $\mathbb{R}$ with $n \geq 2$.

The class of *Sierpiński-Zygmund* functions is independent of all the classes included in the above chart in the following sense. There is no inclusion between SZ and AC, Conn, D, or PC. SZ is disjoint with C and Ext. (See also comment below Corollary 5.) $\mathrm{SZ}(\mathbb{R}^n)$ is disjoint with $\mathrm{D}(\mathbb{R}^n)$ and $\mathrm{AC}(\mathbb{R}^n)$ for $n \geq 2$. (See Remarks 7 and 8.)

The class of additive functions $\mathrm{AD}(\mathbb{R}^n)$ intersects each of the other classes (the non-emptiness of $\mathrm{AD} \cap \mathrm{SZ}$ follows from Theorem 10 (iv) and Proposition 1 (4).) However, it is not contained in any of them except the family $\mathrm{PC}(\mathbb{R}^n)$ in the case $n = 1$. Then we have $\mathrm{AD} \subseteq \mathrm{PC}$.

Now let us comment on $\mathrm{A}(\mathcal{F})$ for $\mathcal{F} \in \{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}, \mathrm{SZ}\}$. The following can be proved in ZFC:

$$\mathfrak{c}^+ = \mathrm{A}(\mathrm{Ext}) \leq \mathrm{A}(\mathrm{AC}) = \mathrm{A}(\mathrm{Conn}) = \mathrm{A}(\mathrm{D}) \leq \mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}},$$

$$\mathfrak{c}^+ \leq \mathrm{A}(\mathrm{SZ}) \leq 2^{\mathfrak{c}}.$$

For more details see [4], [5], [6], and [13].

The main result of the paper is the following theorem.

Theorem 2.

(1) (MA) $\mathrm{Add}(\mathrm{D}, \mathrm{SZ}) \geq \mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) \geq \omega$.

(2) (MA) $\mathrm{Add}(\mathrm{SZ}, \mathrm{AC}) = \mathrm{Add}(\mathrm{SZ}, \mathrm{D}) = \mathfrak{c}$.

(3) *If the theory "ZFC + $\exists$ measurable cardinal" is consistent then so is "ZFC + $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) > \mathfrak{c} > \omega_1$."*

(4) $\mathrm{Add}(\mathrm{PC}, \mathrm{SZ}) = \mathrm{A}(\mathrm{SZ})$ *and* $\mathrm{Add}(\mathrm{SZ}, \mathrm{PC}) = 2^{\mathfrak{c}}$.

The following remains an open problem. (See Fact 15.)

Problem 3 Does the equality $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) = \omega$ hold in "ZFC + MA" (or in "ZFC + CH"?)

Let us make here some comments about the theorem. Parts (1) and (3) give only lower bound for $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ})$. So one may wonder whether it is possible to give in ZFC any non-trivial upper bound for that number. However, in the model used to prove (3) it is possible to have $\mathfrak{c}^+ = 2^{\mathfrak{c}}$, so it cannot be proved in ZFC that $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) < 2^{\mathfrak{c}}$. But it is unknown whether $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) \leq \mathfrak{c}^+$ in ZFC. The next comment is about symmetry of Add. It is consistent that $\mathrm{A}(\mathrm{SZ}) < 2^{\mathfrak{c}}$. (See [5].) Hence the part (4) implies that Add is not symmetric in general.

Next we give some corollaries of the main result. To state the first one, note that $-\mathrm{SZ} = \{-f : f \in \mathrm{SZ}\} = \mathrm{SZ}$. This observation, Proposition 1 and the part (2) of Theorem 2 immediately imply the following corollary.

Corollary 4 (MA) *Every function $f : \mathbb{R} \to \mathbb{R}$ can be represented as a sum of almost continuous and Sierpiński-Zygmund functions.*

Let us mention that the corollary, so also the parts (1) and (2) of Theorem 2, cannot be proved in ZFC alone (i.e., without any additional assumptions.) Indeed, if $\mathbb{R}^{\mathbb{R}} = \mathrm{AC} + \mathrm{SZ}$ then there exists an almost continuous function which is also Sierpiński-Zygmund. An example of a model with no Darboux (so also almost continuous) Sierpiński-Zygmund function is given in [1]. Hence we can state

Corollary 5 *The equalities $\mathbb{R}^{\mathbb{R}} = \mathrm{AC} + \mathrm{SZ}$ and $\mathbb{R}^{\mathbb{R}} = \mathrm{D} + \mathrm{SZ}$ are independent of ZFC.*

One may ask whether Corollary 4 can be improved by replacing the family AC of almost continuous functions by the family Ext of extendable functions. However, it cannot be done. The reason is that every extendable function is continuous on some perfect set. (See [3].) The above observation implies

Fact 6 $\mathrm{Add}(\mathrm{Ext}, \mathrm{SZ}) = \mathrm{Add}(\mathrm{SZ}, \mathrm{Ext}) = 1$.

One may also try to generalize Corollary 4 for all functions from $\mathbb{R}^n$ into $\mathbb{R}$. However, in the case $n \geq 2$ it can be proved in ZFC that there is no almost continuous function which is also Sierpiński-Zygmund. We have the following remark.

Remark 7 Let $n \geq 2$. Then $\mathrm{AC}(\mathbb{R}^n) \cap \mathrm{SZ}(\mathbb{R}^n) = \emptyset$ and

$$\mathrm{Add}(\mathrm{AC}(\mathbb{R}^n), \mathrm{SZ}(\mathbb{R}^n)) = \mathrm{Add}(\mathrm{SZ}(\mathbb{R}^n), \mathrm{AC}(\mathbb{R}^n)) = 1.$$

Proof. For every $n \geq 2$ if $f \in \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{SZ}(\mathbb{R}^n)$ then $f|\mathbb{R}^2 \in \mathrm{AC}(\mathbb{R}^2) \cap \mathrm{SZ}(\mathbb{R}^2)$. (See [13].) Hence it is enough to prove the remark for $n = 2$. We construct the family $\{B_y : y \in \mathbb{R}\}$ of $\mathfrak{c}$-many blocking sets in $\mathbb{R}^3$ with pairwise disjoint xy-projections and whose union is the graph of a continuous function. Let $B_y = \{\langle x, y, \tan(x)\rangle : x \in (\frac{-\pi}{2}, \frac{\pi}{2})\}$ for $y \in \mathbb{R}$. Every almost continuous function from $\mathbb{R}^2$ to $\mathbb{R}$ must intersect all sets B_y. Thus it cannot be of Sierpiński-Zygmund type, since it agrees with the function $F(x, y) = \tan(x)$ on a set of cardinality of continuum.

The second part of the conclusion follows from Proposition 1 (4). ■

Let us make here a comment about $\mathrm{Add}(\mathrm{D}(\mathbb{R}^n), \mathrm{SZ}(\mathbb{R}^n))$. It is easy to see that $\mathrm{SZ}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) = \emptyset$ because for each non-constant Darboux function $f\colon \mathbb{R}^n \to \mathbb{R}$ there exists a real number y such that $f^{-1}(y)$ disconnects $\mathbb{R}^n$. Based on this we obtain

Remark 8 $\mathrm{Add}(\mathrm{D}(\mathbb{R}^n), \mathrm{SZ}(\mathbb{R}^n)) = \mathrm{Add}(\mathrm{SZ}(\mathbb{R}^n), \mathrm{D}(\mathbb{R}^n)) = 1$.

The next two theorems describe the function Add for other pairs of classes considered in this paper.

Theorem 9. *Let* $\mathcal{F} \in \{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}\}$ *and* $\mathcal{F}_1, \mathcal{F}_2 \in \{\mathrm{AC}, \mathrm{Conn}, \mathrm{D}\}$. *The following equalities hold.*

(i) $\mathrm{Add}(\mathrm{C}, \mathcal{F}) = \mathrm{Add}(\mathcal{F}, \mathrm{C}) = 1$.

(ii) $\mathrm{Add}(\mathcal{F}, \mathrm{Ext}) = \mathrm{A}(\mathrm{Ext}) = \mathfrak{c}^+$ *and* $\mathrm{Add}(\mathrm{Ext}, \mathcal{F}) = \mathrm{A}(\mathcal{F})$.

(iii) $\mathrm{Add}(\mathcal{F}, \mathrm{PC}) = \mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$.

(iv) $\mathrm{Add}(\mathcal{F}_1, \mathcal{F}_2) = \mathrm{A}(\mathrm{D})$.

Theorem 10. *Let* $\mathcal{F} \in \{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}, \mathrm{SZ}\}$. *The following holds.*

(i) $\mathrm{Add}(\mathrm{AD}, \mathrm{AC}) = \mathrm{Add}(\mathrm{AD}, \mathrm{Conn}) = \mathrm{Add}(\mathrm{AD}, \mathrm{D}) = \mathrm{A}(\mathrm{AC})$.

(ii) $\mathrm{Add}(\mathrm{AD}, \mathrm{Ext}) = \mathrm{A}(\mathrm{Ext}) = \mathfrak{c}^+$.

(iii) $\mathrm{Add}(\mathrm{AD}, \mathrm{PC}) = \mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$.

(iv) $\mathrm{Add}(\mathrm{AD}, \mathrm{SZ}) > \mathfrak{c}$.

(v) $\mathrm{Add}(\mathcal{F}, \mathrm{AD}) = \mathrm{A}(\mathrm{AD}) = 2$ *and* $\mathrm{Add}(\mathrm{C}, \mathrm{AD}) = \mathrm{Add}(\mathrm{AD}, \mathrm{C}) = 1$.

We state here next open problem.

Problem 11 Does $\mathrm{Add}(\mathrm{AD}, \mathrm{SZ})$ equal to $\mathrm{A}(\mathrm{SZ})$?

The paper is organized as follows. The proof of Theorem 2 is presented in next three sections. The proof of parts (1)-(2) is given in Section 2. It is based on two auxiliary results (Lemmas 12 and 13) which are of interest on their own. The proofs of parts (3) and (4) are presented in Sections 3 and 4, respectively. In Section 5 we prove Theorems 9 and 10.

2 Proof of Theorem 2 (1)-(2)

We begin this section with presenting two lemmas. To state the lemmas we need the following definitions. For $X \subseteq \mathbb{R}$ by $\mathrm{C}^{<\mathfrak{c}}(X)$ we denote the family of all functions $f\colon X \to \mathbb{R}$ which can be represented as a union of less than $\mathfrak{c}$-many partial continuous functions. To simplify notation we write $\mathrm{C}^{<\mathfrak{c}}$ and $\mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}}$ for $\mathrm{C}^{<\mathfrak{c}}(\mathbb{R})$ and $\bigcup_{X\subseteq\mathbb{R}} \mathrm{C}^{<\mathfrak{c}}(X)$, respectively. Observe that under the assumption of regularity of $\mathfrak{c}$ (so also under MA) $\mathrm{SZ}(X) + \mathrm{C}^{<\mathfrak{c}}(X) = \mathrm{SZ}(X)$ and $\mathrm{SZ}(Y) \cap \mathrm{C}^{<\mathfrak{c}}(Y) = \emptyset$ for any $X, Y \subseteq \mathbb{R}$ with $|Y| = \mathfrak{c}$. The same assumption about $\mathfrak{c}$ implies also that the union of any family $F \subseteq \mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}}$ of cardinality less than $\mathfrak{c}$ contains a function from $\mathrm{C}^{<\mathfrak{c}}(\bigcup_{f\in F} \mathrm{dom}(f))$.

Now we introduce the next definition. Let $A \subseteq \mathbb{R}$ be everywhere of second category, that is $A \cap I$ is of second category for every nontrivial interval I. We define $\mathcal{F}_A$ as a family of all $F \subseteq \mathbb{R}^{\mathbb{R}}$ whose union $\bigcup F$ contains no function from $\mathrm{C}^{<\mathfrak{c}}(A \cap B)$ for any non-meager Borel set B. That is

$$\mathcal{F}_A = \left\{ F \subseteq \mathbb{R}^{\mathbb{R}} \colon \forall B \in (\mathcal{B} \setminus \mathcal{M})\ \ \forall f \in \mathrm{C}^{<\mathfrak{c}}(A \cap B)\ \ f \not\subseteq \bigcup F \right\}.$$

Lemma 12 (MA) *Let $F \in \mathcal{F}_A$ be a family such that $|F| < \mathrm{A}(\mathrm{SZ})$. There exists a $g \in \mathrm{SZ}(A)$ such that every extension $\bar{g}\colon \mathbb{R} \to \mathbb{R}$ of g is almost continuous and $g + F \subseteq \mathrm{SZ}(A)$.*

Proof. Let $\langle f_\alpha : \alpha < \mathfrak{c}\rangle$ be a sequence of all continuous functions defined on G_δ subsets of $\mathbb{R}$.

(1) First we construct a partial real function $g' \in \mathrm{SZ}_{\mathrm{part}}$ with $\mathrm{dom}(g') \subseteq A$ such that for every $f \in F$, $g' + f \in \mathrm{SZ}_{\mathrm{part}}$ and any extension of g' on $\mathbb{R}$ is in AC. We do this by transfinite induction. We construct a sequence $\langle g_\xi : \xi < \mathfrak{c}\rangle$ of partial real functions satisfying the following conditions for every $\alpha < \mathfrak{c}$:

(a) $D_\alpha = \mathrm{dom}(g_\alpha)$ is countable;

(b) g_α is dense subset of $(f_\alpha|A) \setminus \bigcup_{\xi<\alpha} (f_\xi \cup (D_\xi \times \mathbb{R}) \cup \bigcup(f_\xi - F))$.

Notice that $D_\alpha \cap D_\beta = \emptyset$ and $D_\alpha \subseteq A$ for $\alpha < \beta < \mathfrak{c}$. Now we define $g' = \bigcup_{\xi<\mathfrak{c}} g_\xi$. We will show that g' has the required properties.

(i) $g', g' + f \in \mathrm{SZ}_{\mathrm{part}}$, for every $f \in F$.

Let $\xi < \mathfrak{c}$. We see from the condition (b) that $[g' = f_\xi]$, $[(g' + f) = f_\xi] \subseteq \bigcup_{\alpha\leq\xi} D_\alpha$. Hence $|[g' = f_\xi]|, |[(g' + f) = f_\xi]| \leq \xi\omega < \mathfrak{c}$.

(ii) Any extension of g' is an almost continuous function.

We will prove that g' intersects every blocking set $B \subseteq \mathbb{R}$. B contains a continuous function q defined on a Borel set of second category. (See [11].) Let α_B be the smallest ordinal number such that f_{α_B} agrees with q on a set residual in some interval $J \subseteq \mathrm{dom}(B)$. B is closed and therefore

$f_{\alpha_B}|J \subseteq B$. From the definition of α_B and MA we see that $\bigcup_{\xi<\alpha_B}[f_\xi = q]$ is of first category as the union of less than $\mathfrak{c}$-many sets of first category. Recall that $F \in \mathcal{F}_A$. This implies that $(I\cap A)\setminus\bigcup_{\xi<\alpha_B}\bigcup_{f\in F}[(f_\xi - f) = q]$ is of second category for every nontrivial interval I. The above holds because otherwise we would have that $(K \cap A) \subseteq \bigcup_{\xi<\alpha_B}\bigcup_{f\in F}[(f_\xi - f) = q]$ for some $K \in \mathcal{B}\setminus\mathcal{M}$. Then for every $x \in (K \cap A)$ there are $\xi < \alpha_B$ and $f \in F$ such that $f_\xi(x) - f(x) = q(x)$. Define $h\colon (K \cap A) \to \mathbb{R}$ by $h(x) = f_\xi(x) - q(x) = f(x)$. It is easy to see that h is a subset of both $\bigcup_{\xi<\alpha_B}(f_\xi - q)$ and $\bigcup F$. In particular, it implies that $h \in \mathrm{C}^{<\mathfrak{c}}(K \cap A)$ which contradicts the assumption that $F \in \mathcal{F}_A$.

Hence $(J \cap A) \setminus \bigcup_{\xi<\alpha_B}(\bigcup_{f\in F}[(f_\xi - f) = q] \cup [f_\xi = q] \cup D_\xi)$ is of second category. Therefore $D_{\alpha_B} \cap J \neq \emptyset$. This implies $g' \cap B \supseteq g_{\alpha_B} \cap B \neq \emptyset$ (g_{α_B} and f_{α_B} coincide on $D_{\alpha_B} \cap J$).

(2) Let $g'' : A \setminus \mathrm{dom}(g') \to \mathbb{R}$ be a Sierpiński-Zygmund function such that $g'' + F \subseteq \mathrm{SZ}_{\mathrm{part}}$. Such a function exists because $|F| < \mathrm{A(SZ)}$. We define $g = g' \cup g''$. We see that $g \in \mathrm{SZ}(A)$, any extension of g onto $\mathbb{R}$ is in AC, and $g + F \subseteq \mathrm{SZ}(A)$. ∎

Lemma 13 (MA) *Let $\{f_i\}_1^n \subseteq \mathbb{R}^\mathbb{R}$, $n = 1, 2, \ldots$. There exists $\{f'_i\}_1^n \in \mathcal{F}_A$ such that $f_i|A_i \in \mathrm{C}^{<\mathfrak{c}}(A_i)$, where $A_i = [f_i \neq f'_i]$.*

Proof. The proof is by induction on number n of functions.

Assume that the lemma is true for every $\{g_i\}_1^{n-1} \subseteq \mathbb{R}^\mathbb{R}$, $n \geq 1$. Let us fix $\{f_i\}_1^n \subseteq \mathbb{R}^\mathbb{R}$. We will construct a family $\{f'_i\}_1^n \in \mathcal{F}_A$ such that $f_i|[f_i \neq f'_i] \in \mathrm{C}^{<\mathfrak{c}}([f_i \neq f'_i])$ for all $i \leq n$.

We start with showing that the following claim holds for all $f, h, h' \in \mathbb{R}^\mathbb{R}$.

$$\text{If } f|[f \neq h] \in \mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}} \text{ and } h|[h \neq h'] \in \mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}} \text{ then } f|[f \neq h'] \in \mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}}.$$

This is so because we have that $[f \neq h'] \subseteq [f \neq h] \cup [h \neq h']$ and consequently

$$f|[f \neq h'] \subseteq f|([f \neq h] \cup [h \neq h']) = f|[f \neq h] \;\cup\; f|([h \neq h'] \setminus [f \neq h]) \subseteq$$

$$\subseteq f|[f \neq h] \;\cup\; h|[h \neq h'].$$

This completes the proof of the claim.

Now observe that, by the inductive assumption, there exists $\{h_i\}_2^n \in \mathcal{F}_A$ such that $f_i|[f_i \neq h_i] \in \mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}}$ for $i = 2, \ldots, n$. Put $h_1 = f_1$. If $\{h'_i\}_1^n \in \mathcal{F}_A$ is such that $h_i|[h_i \neq h'_i] \in \mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}}$ for $i = 1, \ldots, n$ then, based on the above claim, also $f_i|[f_i \neq h'_i] \in \mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}}$ for all i. So without loss of generality we may assume that $\{f_i\}_2^n \in \mathcal{F}_A$.

Next we define the family $\mathcal{B}_{f_1,\ldots,f_n}$ as follows

$$\mathcal{B}_{f_1,\ldots,f_n} = \{A \cap B\colon B \in \mathcal{B}\setminus\mathcal{M} \;\&\; \exists f \in \mathrm{C}^{<\mathfrak{c}}(A \cap B)\; f \subseteq \bigcup f_i\}.$$

There exists a maximal element $A_{\max}$ in $\mathcal{B}_{f_1,\dots,f_n}$ with respect to the relation $\subseteq^*$ defined by

$$X_1 \subseteq^* X_2, \text{ if } X_1 \setminus X_2 \text{ is of first category.}$$

To prove the existence let us consider $\mathcal{S} = \{B \in \mathcal{B} \setminus \mathcal{M} \colon A \cap B \in \mathcal{B}_{f_1,\dots,f_n}\}$. For every $B \in \mathcal{S}$ we define a maximal open set U_B such that B is residual in U_B. Since $\mathbb{R}$ has a countable base, there is a sequence $\langle B_n \in \mathcal{S} \colon n < \omega\rangle$ such that $\bigcup_{B\in\mathcal{S}} U_B = \bigcup_{n<\omega} U_{B_n}$. We claim that $A_{\max} = \bigcup_{n<\omega}(A \cap B_n)$ is the desired maximal element. First we notice that $A_{\max} \in \mathcal{B}_{f_1,\dots,f_n}$. Now, let $A \cap B \in \mathcal{B}_{f_1,\dots,f_n}$. From the properties of the sets B_n $(n < \omega)$ we get that $B \subseteq^* U_B \subseteq \bigcup_{n<\omega} U_{B_n} \subseteq^* \bigcup_{n<\omega} B_n$. So $A \cap B \subseteq^* A_{\max}$.

Now, let f be the function associated with $A_{\max}$ (e.g. $f \in \mathrm{C}^{<\mathfrak{c}}(A_{\max})$ and $f \subseteq \bigcup f_i$). The function f can be represented as $f = \bigcup f_i|A_i$, where $\bigcup_{i\le n} A_i = A_{\max}$, $A_i \cap A_j = \emptyset$ $(i \neq j)$, and $f_i|A_i \in \mathrm{C}^{<\mathfrak{c}}(A_i)$. Let us consider the following functions $f_i' = f_i|(\mathbb{R} \setminus A_i) \cup g_i$, where $g_i \in \mathrm{SZ}(A_i)$ $(i = 1,\dots,n)$. We will show that $\{f_i'\}_1^n$ is the required family, that is $\{f_i'\}_1^n \in \mathcal{F}_A$. Assume, by contradiction, that $\{f_i'\}_1^n \notin \mathcal{F}_A$. Thus there exists a set A' of the form $A \cap B$ for some $B \in \mathcal{B} \setminus \mathcal{M}$ such that $A' = \bigcup A_i'$, A_i' are pairwise disjoint and $f_i'|A_i' \in \mathrm{C}^{<\mathfrak{c}}(A_i')$. Let us denote $\bigcup(f_i'|A_i')$ by f'. Note that $A' \subseteq^* A_{\max}$. Since $g_1 \in \mathrm{SZ}(A_1)$, we have $|A_1 \cap A_1'| < \mathfrak{c}$. This observation and Martin's Axiom imply that $A_1 \cap A_1' \in \mathcal{M}$. So we may assume $A_1 \cap A_1' = \emptyset$. Then $f'|(A_1 \cap A') \subseteq \bigcup_{i=2}^n f_i$. This implies that $f'|(A_1 \cap A') \cup f|(\bigcup_{i=2}^n A_i \cap A') \in \mathrm{C}^{<\mathfrak{c}}(A')$. Hence $\bigcup_{i=2}^n f_i$ contains a function from $\mathrm{C}^{<\mathfrak{c}}(A')$. So $\{f_i\}_2^n \notin \mathcal{F}_A$. Contradiction. ∎

Before we show how the above two lemmas imply parts (1) and (2) of the main result, let us make a remark regarding Lemma 13. One could expect the lemma to hold for bigger families of functions. However, Lemma 13 cannot be generalized for infinite families of functions. Let us see the following counterexample.

Example 14 (CH) *There exists an infinite family $\{f_n\}_{n<\omega} \subseteq \mathbb{R}^{\mathbb{R}}$ for which the conclusion of Lemma 13 fails.*

Proof. Continuum Hypothesis implies the existence of an Ulam matrix on $\mathbb{R}$, e.g. the family $\{M_\xi^n : n < \omega, \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}$ with

$$M_\xi^n \cap M_\alpha^n = \emptyset, \text{ for } n < \omega,\ \xi < \alpha < \mathfrak{c},$$

$$\text{the complement of } \bigcup_{n<\omega} M_\xi^n \text{ is a countable set, for } \xi < \mathfrak{c}.$$

Fix an enumeration $\{x_\xi : \xi < \mathfrak{c}\}$ of $\mathbb{R}$. Define f_n as an extension of $\bigcup_{\xi<\mathfrak{c}} x_\xi \chi_{M_\xi^n}$ onto $\mathbb{R}$, for every $n < \omega$. We are now in a position to show that $F = \{f_n \colon n < \omega\}$ is the counterexample for the conclusion of Lemma 13. Since every vertical section of $\bigcup F$ is countable and every horizontal section is comeager, it follows that $\bigcup F$ is non-Borel set of second category. Now, let $A_n \subseteq \mathbb{R}$ be such that $f_n|A_n \in \mathrm{CC}(A_n)$, for every n. Since the graph of a continuous function is meager in $\mathbb{R}^2$, we obtain that $\bigcup_{n<\omega} f_n|A_n$ is also meager as a union of countably many meager sets. We conclude from this that there exists a meager horizontal section of $\bigcup_{n<\omega} f_n|A_n$. Therefore the set $\bigcup F \setminus \bigcup_{n<\omega} f_n|A_n$ contains a constant function defined on comeager Borel set. ■

Using very similar technique as the above we can prove

Fact 15 (CH) Either $\mathrm{Add}(\mathrm{AC},\mathrm{SZ}) = \omega$ or $\mathrm{Add}(\mathrm{AC},\mathrm{SZ}) > \mathfrak{c}$.

Proof. Let us assume that $F = \{\phi_\xi \colon \xi < \mathfrak{c}\} \subseteq \mathbb{R}^\mathbb{R}$ witnesses $\mathrm{Add}(\mathrm{AC},\mathrm{SZ}) \leq \mathfrak{c}$. For every $n < \omega$, define a function f_n^* as an extention of $\bigcup_{\xi<\mathfrak{c}} \phi_\xi \chi_{M_\xi^n}$ onto $\mathbb{R}$, where $\{M_\xi^n : n < \omega, \xi < \mathfrak{c}\}$ is an Ulam matrix. We claim that $\{f_n^* \colon n < \omega\}$ witnesses $\mathrm{Add}(\mathrm{AC},\mathrm{SZ}) \leq \omega$. To see this fix an $h \in \mathrm{AC}$. By our assumption about F, there exists an $\xi_0 < \mathfrak{c}$ such that $h + f_{\xi_0} \notin \mathrm{SZ}$. That means $h + f_{\xi_0}$ is continuous on a set X of cardinality continuum. Since $\mathbb{R} \setminus \bigcup_{n<\omega} M_{\xi_0}^n$ is countable we obtain that $|X \cap M_{\xi_0}^m| = \mathfrak{c}$ for some $m < \omega$. Hence $h + f_m^*$ is continuous on a set of cardinality continuum which means that $h + f_m^* \notin \mathrm{SZ}$. ■

Proof of $\mathrm{Add}(\mathrm{AC},\mathrm{SZ}) \geq \omega$ (under MA).

We begin by fixing $F = \{f_1, \ldots, f_n\} \subseteq \mathbb{R}^\mathbb{R}$. Let $F' = \{f'_1, \ldots, f'_n\} \in \mathcal{F}_\mathbb{R}$ be a corresponding family given by Lemma 13 for $A = \mathbb{R}$. Based on Lemma 12, we can find a $g \in \mathrm{AC} \cap \mathrm{SZ}$ such that $g + F' \subseteq \mathrm{SZ}$. Since $f_i|[f'_i \neq f_i] \in \mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}}$ and $g \in \mathrm{SZ}$, we obtain that $g + f_i \in \mathrm{SZ}$ (for $i = 1, 2, \ldots,$ n.) ■

In order to prove part (2) of Theorem 2 we need to state one more lemma.

Lemma 16 $\mathrm{Add}(\mathrm{SZ},\mathrm{D}) \leq 2^{<\mathfrak{c}}$.

Proof. Let us consider the following family of functions $\mathcal{F}^{<\mathfrak{c}} = \{r\chi_A \colon A \in [\mathbb{R}]^{<\mathfrak{c}}, r \in \mathbb{Q}\}$. Obviously $|\mathcal{F}^{<\mathfrak{c}}| = 2^{<\mathfrak{c}}$. We claim that

$$\forall_{g \in SZ}\ g + \mathcal{F}^{<\mathfrak{c}} \not\subseteq \mathrm{D}.$$

To see this, fix $g \in \mathrm{SZ}$. Let $r_0 \in \mathbb{Q}$ such that $\inf g < r_0 < \sup g$. Then $g - r_0\chi_A \notin \mathrm{D}$, where $A = g^{-1}[r_0]$. ■

Proof of $\mathrm{Add}(\mathrm{SZ},\mathrm{AC}) = \mathrm{Add}(\mathrm{SZ},\mathrm{D}) = \mathfrak{c}$ (under MA).

Since $\mathrm{Add}(\mathrm{SZ},\mathrm{AC}) \leq \mathrm{Add}(\mathrm{SZ},\mathrm{D})$ and $\mathrm{Add}(\mathrm{SZ},\mathrm{D}) \leq 2^{<\mathfrak{c}} = \mathfrak{c}$ (assuming MA), it is sufficient to prove that for every family $F \subseteq \mathbb{R}^\mathbb{R}$ of cardinality less than $\mathfrak{c}$ there exists a Sierpiński-Zygmund function $h \colon \mathbb{R} \to \mathbb{R}$ satisfying the property $h + F \subseteq \mathrm{AC}$.

Let $F = \{f_\xi \colon \xi < \kappa\} \subseteq \mathbb{R}^\mathbb{R}$ ($\kappa = |F| < \mathfrak{c}$) and $\{A_\xi \colon \xi < \kappa\}$ be a partition of $\mathbb{R}$ into Bernstein sets. By Lemma 13, for every $\xi < \kappa$ we can find a function f'_ξ such that the singleton $\{f'_\xi\}$ belongs to $\mathcal{F}_{A_\xi}$ and $f'_\xi|[f'_\xi \neq f_\xi] \in \mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}}$. Now, applying Lemma 12 for every $\xi < \kappa$ we obtain a sequence $\langle g_\xi \colon A_\xi \to \mathbb{R} : \xi < \kappa\rangle$ for which the following holds

$$g_\xi + f'_\xi \in \mathrm{SZ}_{\mathrm{part}} \text{ and any extension of } g_\xi \text{ on } \mathbb{R} \text{ is in AC, for } \xi < \kappa.$$

Since $f'_\xi|[f'_\xi \neq f_\xi] \in \mathrm{C}^{<\mathfrak{c}}_{\mathrm{part}}$ and $\mathrm{SZ}(X) + \mathrm{C}^{<\mathfrak{c}}(X) = \mathrm{SZ}(X)$ for every $X \subseteq \mathbb{R}$, we conclude that $g_\xi + f_\xi \in \mathrm{SZ}_{\mathrm{part}}$, $\xi < \kappa$. Put $h = \bigcup_{\xi<\kappa} -(g_\xi + f_\xi)$. Since Martin's Axiom implies the regularity of $\mathfrak{c}$ we obtain that $h \in \mathrm{SZ}$. Clearly, $h + F \subseteq \mathrm{AC}$. ∎

As the final remark let us notice that parts (1) and (2) of the main result as well as Lemmas 12 and 13 could be proved under weaker assumptions. The proofs require only two consequences of Martin's Axiom: $\mathfrak{c} = \mathfrak{c}^{<\mathfrak{c}}$ (this implies regularity of $\mathfrak{c}$); the union of less than $\mathfrak{c}$-many meager sets is meager.

3 Proof of Theorem 2 (3)

We will show that the existence of $\mathfrak{c}$-additive σ-saturated ideal $\mathcal{J}$ in $P(\mathbb{R})$ containing $\mathcal{M}$ implies $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) > \mathfrak{c}$. It is known that the existence of such an ideal is equiconsistent with "ZFC + $\exists$ measurable cardinal."[2] (See [9].)

First notice that we may assume that $\mathcal{J} \cap \mathcal{B} = \mathcal{M}$. To see this suppose that there exists a Borel set B of second category in $\mathcal{J}$. B is residual in some open interval I. Then $I \in \mathcal{J}$ because $I \setminus B$ is meager and $I = (B \cap I) \cup (I \setminus B)$. Now, let U be a maximal open set belonging to $\mathcal{J}$. Such a set exists because the union of all open sets from $\mathcal{J}$ can be represented as a union of countable many such sets. We have that $\mathbb{R} \setminus U$ contains a nonempty open interval I_0. Otherwise it would be nowhere-dense and then $\mathbb{R} = U \cup (\mathbb{R} \setminus U) \in \mathcal{J}$. Now, any homeomorphism between I_0 and $\mathbb{R}$ induces the desired ideal on $\mathbb{R}$.

The schema of the proof is similar to the idea of combining Lemmas 12 and 13 in the proof of $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) \geq \omega$. First step is to show that

$(*)$ for each $f\colon \mathbb{R} \to \mathbb{R}$ there exists an $f^{\mathcal{J}} \in \mathbb{R}^{\mathbb{R}}$ such that $f|[f \neq f^{\mathcal{J}}] \in \mathrm{CC}_{\mathrm{part}}$ and $f^{\mathcal{J}}|X \notin \mathrm{CC}(X)$ for every $X \notin \mathcal{J}$.

To see this fix an $f \in \mathbb{R}^{\mathbb{R}}$. We claim that there exists a set Y such that $f|Y \in \mathrm{CC}(Y)$ and $Y' \subseteq^{\mathcal{J}} Y$ for all Y' satisfying $f|Y' \in \mathrm{CC}(Y')$, where $\subseteq^{\mathcal{J}}$ is defined by

$$Z_1 \subseteq^{\mathcal{J}} Z_2, \text{ if } Z_1 \setminus Z_2 \in \mathcal{J}.$$

If the claim did not hold then we could easily construct a strictly increasing (in terms of $\subseteq^{\mathcal{J}}$) uncountable sequence of subsets of $\mathbb{R}$. Indeed, assume that the desired sequence of sets X_ξ is defined for all $\xi < \alpha$, where $\alpha < \omega_1$. Note that $f|\bigcup_{\xi<\alpha} X_\xi \in \mathrm{CC}_{\mathrm{part}}$. By assumption there exists a set X such that $\bigcup_{\xi<\alpha} X_\xi \subseteq^{\mathcal{J}} X \not\subseteq^{\mathcal{J}} \bigcup_{\xi<\alpha} X_\xi$ and $f|X \in \mathrm{CC}_{\mathrm{part}}$. We set $X_\alpha = X$. Thus by transfinite induction the sequence is defined for all $\alpha < \omega_1$. But the existence of this sequence would imply the existence of an uncountable family of disjoint sets outside of $\mathcal{J}$ which contradicts the fact that $\mathcal{J}$ is σ-saturated.

So we proved that the set Y exists. Now put $f^{\mathcal{J}} = f|(\mathbb{R} \setminus Y) \cup g$, where g is any function from $\mathrm{SZ}(Y)$. Clearly, $f^{\mathcal{J}}$ is the desired function from $(*)$.

[2] The desired model is obtained by adding κ-many Cohen reals, where κ is a measurable cardinal in the ground model.

In the next step we fix a family F of real functions of cardinality $\mathfrak{c}$. Let $F = \{h_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of F and $\langle f_\alpha : \alpha < \mathfrak{c}\rangle$ be a sequence of all continuous functions defined on G_δ subsets of $\mathbb{R}$. Based on the previous reasoning we may assume that $h_\xi|X \notin \mathrm{CC}(X)$ for every $X \notin \mathcal{J}$ and $\xi < \mathfrak{c}$. Notice that if $\gamma, \alpha < \mathfrak{c}$ and $f_\alpha|X \subseteq \bigcup_{\xi,\beta<\gamma}(f_\xi - h_\beta)$ then $X \in \mathcal{J}$. This is so since $X \subseteq \bigcup_{\xi,\beta<\gamma}[f_\alpha = f_\xi - h_\beta]$ and every set $[f_\alpha = f_\xi - h_\beta] = [h_\beta = f_\xi - f_\alpha] \in \mathcal{J}$. Consequently, the set $\mathrm{dom}(f_\alpha \setminus \bigcup_{\xi,\gamma<\alpha}(f_\xi - h_\gamma))$ does not belong to $\mathcal{J}$ provided $\mathrm{dom}(f_\alpha) \notin \mathcal{J}$.

Now we construct a sequence $\langle g_\xi : \xi < \mathfrak{c}\rangle$ of partial functions such that

$$g_\alpha \text{ is a countable dense subset of } f_\alpha \setminus \bigcup_{\xi,\gamma<\alpha} ((f_\xi - h_\gamma) \cup f_\xi \cup L(D_\xi)) \text{ for } \alpha < \mathfrak{c},$$

where $D_\gamma = \mathrm{dom}(g_\gamma)$.

The same kind of argument as in the proof of Lemma 12 (i)&(ii) shows that $g' = \bigcup_{\xi<\mathfrak{c}} g_\xi$ is in $\mathrm{SZ}_{\mathrm{part}}$ and intersects every blocking set. So if g is any Sierpiński-Zygmund extension of g' then $g \in \mathrm{AC}$ and $g + F \subseteq \mathrm{SZ}$. ■

4 Proof of Theorem 2 (4)

First we prove $\mathrm{Add}(\mathrm{PC}, \mathrm{SZ}) = \mathrm{A}(\mathrm{SZ})$. In order to do it we need the following straightforward lemma.

Lemma 17 *For every function $f \in \mathbb{R}^{\mathbb{R}}$ there is a function $f' \in \mathrm{PC}$ such that $|[f \neq f']| \leq \omega$.*

Proof. Let $g \colon \mathbb{Q} \to \mathbb{Q}$ be a function with dense graph. Then $f' = g \cup f|(\mathbb{R} \setminus \mathbb{Q})$ is the required function. ■

Now, to show $\mathrm{Add}(\mathrm{PC}, \mathrm{SZ}) = \mathrm{A}(\mathrm{SZ})$, let us notice that $\mathrm{Add}(\mathrm{PC}, \mathrm{SZ}) \leq \mathrm{Add}(\mathbb{R}^{\mathbb{R}}, \mathrm{SZ}) = \mathrm{A}(\mathrm{SZ})$. What is left to prove is that $\mathrm{Add}(\mathrm{PC}, \mathrm{SZ}) \geq \mathrm{A}(\mathrm{SZ})$. Let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be a family of cardinality less than $\mathrm{A}(\mathrm{SZ})$. So there exists a function $g \in \mathbb{R}^{\mathbb{R}}$ such that $g + F \subseteq \mathrm{SZ}$. Let $g' \in \mathrm{PC}$ be a function obtained from g by applying Lemma 17. Since every Sierpiński-Zygmund function modified on a set of cardinality less than $\mathfrak{c}$ remains Sierpiński-Zygmund, it is easy to see that $g' + F \subseteq \mathrm{SZ}$.

Before we start proving that $\mathrm{Add}(\mathrm{SZ}, \mathrm{PC}) = 2^{\mathfrak{c}}$, we introduce the following

Definition 18 A set $X \subseteq \mathbb{R}^2$ is called *Sierpiński-Zygmund* set (shortly *SZ-set*), if for every partial real continuous function f we have $|f \cap X| < \mathfrak{c}$.

An argument, similar to the one used in proving the existence of Sierpiński-Zygmund function, leads to

Lemma 19 *There exists an SZ-set $X \subseteq \mathbb{R}^2$ such that $|\mathbb{R} \setminus X_x| < \mathfrak{c}$ for every $x \in \mathbb{R}$, where $X_x = \{y \in \mathbb{R} \colon \langle x, y\rangle \in X\}$.*

Proof. Let $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ and $\langle f_\alpha : \alpha < \mathfrak{c}\rangle$ be the sequences of all real numbers and all continuous functions defined on a G_δ subset of $\mathbb{R}$, respectively. We will define the set X by defining its vertical sections by transfinite induction. For every $\alpha < \mathfrak{c}$ we put

$$X_{x_\alpha} = \mathbb{R} \setminus \{f_\xi(x_\alpha) \colon \xi < \alpha\}.$$

Put $X = \bigcup_{\alpha<\mathfrak{c}} \{x_\alpha\} \times X_{x_\alpha}$. It is obvious that X has the required properties. ■

Corollary 20 *There exists a family $\{Q_x \subseteq \mathbb{R} \colon x \in \mathbb{R}\}$ of pairwise disjoint countable dense sets such that $\bigcup \prod_{x\in\mathbb{R}} Q_x$ is an SZ-set.*

The next lemma is proved in [6].

Lemma 21 [6, Lemma 2.2] *If $B \subseteq \mathbb{R}$ has cardinality $\mathfrak{c}$ and $H \subseteq \mathbb{Q}^B$ is such that $|H| < 2^{\mathfrak{c}}$ then there is a $g \in \mathbb{Q}^B$ such that $h \cap g \neq \emptyset$ for every $h \in H$.*

We give more general version of this lemma.

Lemma 22 *If $B \subseteq \mathbb{R}$ has cardinality $\mathfrak{c}$ and $H \subseteq \prod_{x\in B} Q_x$ is such that $|H| < 2^{\mathfrak{c}}$ then there is a $g \in \prod_{x\in B} Q_x$ such that $h \cap g \neq \emptyset$ for every $h \in H$.*

Proof. For every $x \in B$ let $f_x \colon Q_x \to \mathbb{Q}$ be a bijection. Now, for each $h \in H$ we define h' as follows

$$h'(x) = f_x(h(x)) \text{ for all } x \in B.$$

The family $H' = \{h' \colon h \in H\} \subseteq \mathbb{Q}^B$ has cardinality less than $2^{\mathfrak{c}}$. Thus, by Lemma 21, there is a function $g' \in \mathbb{Q}^B$ intersecting every element of H'. Put $g(x) = f_x^{-1}(g'(x))$, for all $x \in B$. It is clear that $g \in \prod_{x\in B} Q_x$ and $h \cap g \neq \emptyset$ for every $h \in H$. ■

Proof of $\mathrm{Add}(\mathrm{SZ}, \mathrm{PC}) = 2^{\mathfrak{c}}$**.**

The proof follows the idea of the proof of [6, Theorem 1.7 (3)]. Let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be such that $|F| < 2^{\mathfrak{c}}$. We will find a $g \in \mathrm{SZ}$ such that $g + F \subseteq \mathrm{PC}$.

Let $\mathcal{G}$ be the family of all triples $\langle I, p, m\rangle$ where I is a nonempty open interval with rational end-points, $p \in \mathbb{Q}$, and $m < \omega$. For each $\langle I, p, m\rangle \in \mathcal{G}$ define a set $B_{\langle I,p,m\rangle} \subseteq I$ of size $\mathfrak{c}$ such that $B_{\langle I,p,m\rangle} \cap B_{\langle J,q,n\rangle} = \emptyset$ for any distinct $\langle I, p, m\rangle$ and $\langle J, q, n\rangle$ from $\mathcal{G}$.

Let $\langle I, p, m\rangle \in \mathcal{G}$ be fixed. For each $f \in F$ choose $h^f_{\langle I,p,m\rangle} \in \prod_{x\in B_{\langle I,p,m\rangle}} Q_x$ such that

$$\left| p - \left(f(x) + h^f_{\langle I,p,m\rangle}(x) \right) \right| < \frac{1}{m} \text{ for every } x \in B_{\langle I,p,m\rangle}.$$

Then, by Lemma 21 used with a set $H_{\langle I,p,m\rangle} = \left\{ h^f_{\langle I,p,m\rangle} \colon f \in F \right\}$, there exists a $g_{\langle I,p,m\rangle} \in \prod_{x\in B_{\langle I,p,m\rangle}} Q_x$ such that

$$\forall f \in F\ \exists x \in B_{\langle I,p,m\rangle}\ h^f_{\langle I,p,m\rangle}(x) = g_{\langle I,p,m\rangle}(x).$$

Now, let $g \in \prod_{x\in\mathbb{R}} Q_x$ be a common extension of all functions $g_{\langle I,p,m\rangle}$. Corollary 20 implies that g is of Sierpiński-Zygmund type. The function g has also the following property. For every $\langle I,p,m\rangle \in \mathcal{G}$ and every $f \in F$ there exists $x \in B_{\langle I,p,m\rangle} \subseteq I$ such that

$$|p - (f(x) + g(x))| < \frac{1}{m}.$$

So, each function $f + g$, for $f \in F$, is dense in $\mathbb{R}^2$. Thus $f + g \in \mathrm{PC}$. ■

5 Proofs of Theorems 9 and 10

In this section we present proofs of Theorems 9 and 10. Before we do this, let us recall some definitions and cite some theorems. Let $h \in \mathrm{Ext}$. We say that a set $G \subset \mathbb{R}$ is *h-negligible* provided $f \in \mathrm{Ext}$ for every function $f \colon \mathbb{R} \to \mathbb{R}$ for which $f = h$ on a set $\mathbb{R} \setminus G$. For a cardinal number $\kappa \le \mathfrak{c}$, a function $f \colon \mathbb{R} \to \mathbb{R}$ is called *κ strongly Darboux* if $f^{-1}(y)$ is κ-dense. If $\kappa = \omega$ then we simply say that f is strongly Darboux. We denote the family of all κ strongly Darboux functions by $\mathrm{D}(\kappa)$. It is obvious from the definition that

$$\mathrm{D}(\lambda) \subseteq \mathrm{D}(\kappa) \ \text{ for all cardinals } \ \kappa \le \lambda \le \mathfrak{c}.$$

We also introduce the family $\mathrm{D}(\mathrm{P})$ of *perfectly Darboux* functions as the class of all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that $Q \cap f^{-1}(y) \neq \emptyset$ for every perfect set $Q \subseteq \mathbb{R}$ and $y \in \mathbb{R}$. In other words, a function f is perfectly Darboux if for every $y \in \mathbb{R}$ $f^{-1}(y)$ is a Bernstein set. Notice that $\mathrm{D}(\mathrm{P}) \subseteq \mathrm{D}(\kappa)$ for every $\kappa \le \mathfrak{c}$.

The following theorem is proved in [4].

Theorem 23. $\mathrm{A}(\mathrm{AC}) = \mathrm{A}(\mathrm{D}) = \mathrm{A}(\mathrm{D}(\omega_1))$.

A little modification of the proof of the above theorem gives the following lemma.

Lemma 24 *Let* $\mathcal{F} \in \{\mathrm{AD}, \mathrm{Ext}\}$. *Then* $\mathrm{Add}(\mathcal{F}, \mathrm{AC}) = \mathrm{Add}(\mathcal{F}, \mathrm{D})$.

The proof of Lemma 24 requires the use of the following lemma and proposition.

Lemma 25 *Let X be any set of cardinality continuum and $F \subseteq \mathbb{R}^X$ satisfies the condition $|F| < \mathrm{A}(\mathrm{D})$. There exists a $g \colon X \to \mathbb{R}$ such that $(g + f)^{-1}(y) \neq \emptyset$ for each $y \in \mathbb{R}$.*

Proof. Let $b \colon \mathbb{R} \to X$ be a bijection. By Theorem 23 and monotonicity of A we have that $\mathrm{A}(\mathrm{D}) = \mathrm{A}(\mathrm{D}(\omega))$. Hence we can find a $g' \colon \mathbb{R} \to \mathbb{R}$ satisfying the property that $g' + (f \circ b) \in \mathrm{D}(\omega)$ for each $f \in F$. Put $g = g' \circ b^{-1}$. Clearly, g is the desired function. ■

Proposition 26 $\mathrm{A}(\mathrm{D}) = \mathrm{A}(\mathrm{D}(\mathrm{P}))$.

PROOF. Fix a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ of cardinality less than $\mathrm{A}(\mathrm{D})$. Next,let $\{B_\xi : \xi < \mathfrak{c}\}$ and $\{P_\xi : \xi < \mathfrak{c}\}$ be a family of pairwise disjoint Bernstein sets and an enumeration of all perfect subsets of $\mathbb{R}$, respectively. We define the sequence $\langle A_\xi : \xi < \mathfrak{c}\rangle$ by $A_\xi = B_\xi \cap P_\xi$. Obviously the sets A_ξ are pairwise disjoint and each one of them has cardinality $\mathfrak{c}$. Applying Lemma 25 for every $\xi < \mathfrak{c}$ separately, we get a sequence of functions $\langle g_\xi : A_\xi \to \mathbb{R} \mid \xi < \mathfrak{c}\rangle$ such that for every $\xi < \mathfrak{c}$ the following holds

$$\forall f \in F \ \ \forall y \in \mathbb{R} \quad (g_\xi + f)^{-1}(y) \neq \emptyset.$$

Now, if $g \in \mathbb{R}^{\mathbb{R}}$ is any extension of $\bigcup_{\xi<\mathfrak{c}} g_\xi$ onto $\mathbb{R}$ then $g + F \subseteq \mathrm{D}(\mathrm{P})$. ■

Proof of Lemma 24.

First we show that

$(**)$ $\mathrm{Add}(\mathcal{F}, \mathcal{F}_0) > \mathfrak{c}$ for $\mathcal{F}_0 \in \{\mathrm{AC}, \mathrm{D}(\omega_1)\}$.

Let us fix a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ with cardinality $\mathfrak{c}$. To prove the case $\mathcal{F} = \mathrm{AD}$ consider a $\mathfrak{c}$-dense Hamel basis H. There exists a partition $\{B_f : f \in F\}$ of H into $\mathfrak{c}$-dense sets. Since the projection of every blocking set in $\mathbb{R}^2$ contains an interval, we can find, for every $f \in F$, a partial function $g_f : B_f \to \mathbb{R}$ such that $g_f + f$ intersects every blocking set in at least ω_1 points. Thus every extension of $g_f + f$ onto $\mathbb{R}$ is almost continuous and ω_1 strongly Darboux. If $g \in \mathbb{R}^{\mathbb{R}}$ is any function containing $\bigcup_{f\in F} g_f$ then $g + F \subseteq \mathrm{AC} \cap \mathrm{D}(\omega_1)$. In particular, we can choose g to be an additive function. Hence $\mathrm{Add}(\mathrm{AD}, \mathcal{F}_0) > \mathfrak{c}$ for $\mathcal{F}_0 \in \{\mathrm{AC}, \mathrm{D}(\omega_1)\}$.

Now consider the case $\mathcal{F} = \mathrm{Ext}$. If $\mathcal{F}_0 = \mathrm{AC}$ then we have the inequality $\mathrm{Add}(\mathrm{Ext}, \mathrm{AC}) \geq \mathrm{Add}(\mathrm{Ext}, \mathrm{Ext}) = \mathrm{A}(\mathrm{Ext}) = \mathfrak{c}^+ > \mathfrak{c}$ which follows from Proposition 1 (2)&(5). Now, let us focus on the case $\mathcal{F}_0 = \mathrm{D}(\omega_1)$. Let $Q \subseteq \mathbb{R}$ be $\mathfrak{c}$-dense meager F_σ-set. Then, according to [3, Proposition 4.3], there exists an extendable function $f : \mathbb{R} \to \mathbb{R}$ such that the set $\mathbb{R} \setminus Q$ is f-negligible. Since $|F| < \mathrm{A}(\mathrm{D}) = \mathrm{A}(\mathrm{D}(\mathrm{P}))$, there exists a function $h \in \mathbb{R}^{\mathbb{R}}$ such that $h + F \subseteq \mathrm{D}(\mathrm{P})$. Notice here that any perfectly Darboux function modified on a meager set is in $\mathrm{D}(\omega_1)$. This implies that the function $g = f|Q \ \cup \ h|(\mathbb{R} \setminus Q)$ shifts F into $\mathrm{D}(\omega_1) \subseteq \mathrm{D}$. Since $Q \subseteq [f = g]$ we have that $g \in \mathrm{Ext}$. Observe also that F could be any family with $|F| < \mathrm{A}(\mathrm{D}) = \mathrm{A}(\mathrm{D}(\mathrm{P}))$. So we actually proved that

$$\mathrm{Add}(\mathrm{Ext}, \mathrm{D}) \geq \mathrm{Add}(\mathrm{Ext}, \mathrm{D}(\omega_1)) \geq \mathrm{A}(\mathrm{D}).$$

This finishes the proof of $(**)$.

Now the argument follows the schema of the proof of Theorem 23.[3] We start with proving the equality $\mathrm{Add}(\mathcal{F}, \mathrm{D}) = \mathrm{Add}(\mathcal{F}, \mathrm{D}(\omega_1))$. Obviously $\mathrm{Add}(\mathcal{F}, \mathrm{D}) \geq \mathrm{Add}(\mathcal{F}, \mathrm{D}(\omega_1))$. To justify the other inequality let $\kappa = \mathrm{Add}(\mathcal{F}, \mathrm{D}(\omega_1))$. By $(**)$ we get that $\kappa > \mathfrak{c}$. We will show that $\kappa \geq \mathrm{Add}(\mathcal{F}, \mathrm{D})$.

[3] For reader's convenience, we include this slight modification of the proof from [4] in this paper.

Consider a family $G \subseteq \mathbb{R}^{\mathbb{R}}$ of cardinality κ witnessing $\kappa = \mathrm{Add}(\mathcal{F}, \mathrm{D}(\omega_1))$. We define a new family $G^* = \{h \in \mathbb{R}^{\mathbb{R}} \colon\ \exists g \in G\ h =^* g\}$, where $h =^* f$ if and only if $|\{x \colon h(x) \neq f(x)\}| \leq \omega$. Notice here that $|G^*| = \kappa$. This is so because $\kappa > \mathfrak{c}$ and for every $f \in \mathbb{R}^{\mathbb{R}}$ the set $\{h \in \mathbb{R}^{\mathbb{R}} \colon h =^* f\}$ has cardinality $\mathfrak{c}$. We claim that G^* witnesses $\kappa \geq \mathrm{Add}(\mathcal{F}, \mathrm{D})$. Indeed, let $f \in \mathcal{F}$. Then, by the choice of G, there exists a $g \in G$ satisfying the following $f + g \notin \mathrm{D}(\omega_1)$. This implies the existence of a non-trivial closed interval I and $y \in \mathbb{R}$ for which $|I \cap (f+g)^{-1}(y)| \leq \omega$. By modification of g on a countable set, we get a function $g^* \in G^*$ with the property that $(f+g^*)[I] \cap (-\infty, y) \neq \emptyset \neq (f+g^*)[I] \cap (y, \infty)$ and $y \notin (f+g^*)[I]$. Therefore $(f+g^*) \notin \mathrm{D}$. This ends the proof of the equality $\mathrm{Add}(\mathcal{F}, \mathrm{D}) = \mathrm{Add}(\mathcal{F}, \mathrm{D}(\omega_1))$.

What remains to show is that $\mathrm{Add}(\mathcal{F}, \mathrm{AC}) = \mathrm{Add}(\mathcal{F}, \mathrm{D}(\omega_1))$. The inequality $\mathrm{Add}(\mathcal{F}, \mathrm{AC}) \leq \mathrm{Add}(\mathcal{F}, \mathrm{D}) = \mathrm{Add}(\mathcal{F}, \mathrm{D}(\omega_1))$ is obvious, so we just need to prove that $\mathrm{Add}(\mathcal{F}, \mathrm{AC}) \geq \mathrm{Add}(\mathcal{F}, \mathrm{D}(\omega_1))$. This time consider $K \subseteq \mathbb{R}^{\mathbb{R}}$ witnessing $\mathrm{Add}(\mathcal{F}, \mathrm{AC}) = \lambda$. We put $K^* = \{g - h_B \colon g \in K \text{ and } B \text{ is a blocking set}\}$, where $h_B \in \mathbb{R}^{\mathbb{R}}$ is a function such that $h_B|\mathrm{dom}(B) \subseteq B$. Clearly $|K^*| = \lambda$ because there are only continuum many blocking sets and $\lambda > \mathfrak{c}$. Let $f \in \mathcal{F}$. Then, by the choice of K, there exist a $g \in K$ and a blocking set B such that $(f+g) \cap B = \emptyset$. In particular,

$$[f + (g - h_B)] \cap (B - h_B) = [(f+g) \cap B] - h_B = \emptyset,$$

where we define $Z - h_B = \{(x, y - h_B(x)) \colon (x,y) \in Z\}$ for any $Z \subseteq \mathbb{R}^2$. From the definition of h_B we have $\mathrm{dom}(B) \times \{0\} \subseteq (B - h_B)$. Thus $[f + (g - h_B)] \cap [\mathrm{dom}(B) \times \{0\}] = \emptyset$. This means that $f + (g - h_B) \notin \mathrm{D}(\omega_1)$, since $\mathrm{dom}(B)$ contains a non-trivial interval. But $g - h_B \in K^*$, so K^* witnesses $\lambda \geq \mathrm{Add}(\mathcal{F}, \mathrm{D}(\omega_1))$. This finishes the proof of $\mathrm{Add}(\mathcal{F}, \mathrm{AC}) = \mathrm{Add}(\mathcal{F}, \mathrm{D}(\omega_1))$ as well as whole Lemma 24. ■

Proof of Theorem 9.

(i) Notice that it is enough to show (i) for $\mathcal{F} = \mathrm{PC}$ since $\mathrm{Add}(\mathrm{C}, \mathcal{F}) \leq \mathrm{Add}(\mathrm{C}, \mathrm{PC})$ by Proposition 1 (1). To see that $\mathrm{Add}(\mathrm{C}, \mathrm{PC}) = \mathrm{Add}(\mathrm{PC}, \mathrm{C}) = 1$ observe that $\mathrm{C} + \mathrm{PC} = \mathrm{PC}$. Therefore, if $f \notin \mathrm{PC}$ then there is no $g \in \mathrm{C}$ such that $g + f \in \mathrm{PC}$.

(ii) The first part follows from the inequality

$$\mathrm{A}(\mathrm{Ext}) \geq \mathrm{Add}(\mathcal{F}, \mathrm{Ext}) \geq \mathrm{Add}(\mathrm{Ext}, \mathrm{Ext}) = \mathrm{A}(\mathrm{Ext}) = \mathfrak{c}^+,$$

where the first equality is implied by Proposition 1 (5).

To see $\mathrm{Add}(\mathrm{Ext}, \mathcal{F}) = \mathrm{A}(\mathcal{F}) = \mathrm{A}(\mathrm{AC})$ for $\mathcal{F} \in \{\mathrm{AC}, \mathrm{Conn}, \mathrm{D}\}$ let us note that, by Lemma 24 and Proposition 1 (2), $\mathrm{Add}(\mathrm{Ext}, \mathrm{AC}) = \mathrm{Add}(\mathrm{Ext}, \mathrm{Conn}) = \mathrm{Add}(\mathrm{Ext}, \mathrm{D})$. Finally, the desired equality follows from $\mathrm{Add}(\mathrm{Ext}, \mathrm{D}) \geq \mathrm{A}(\mathrm{D})$, which is shown in the prove of $(**)$ in Lemma 24.

The proof of the case $\mathrm{Add}(\mathrm{Ext}, \mathrm{PC}) = \mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$ will be given in (iii).

(iii) Again, by the monotonicity of Add, it suffices to show (iii) for $\mathcal{F} = \mathrm{Ext}$. Let $Q \subseteq \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$ be as in the proof of $(**)$ Lemma 24, i.e., Q is $\mathfrak{c}$-dense meager F_σ-set and f is an extendable function such that $\mathbb{R} \setminus Q$ is f-negligible.

Fix a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ of cardinality less than $2^{\mathfrak{c}}$. Now, a small modification in the proof of the equality $\mathrm{Add}(\mathrm{SZ}, \mathrm{PC}) = 2^{\mathfrak{c}}$ in Section 4 (the sets $B_{\langle I,p,m\rangle}$ can be chosen to be subsets of $\mathbb{R} \setminus Q$), gives us a function $g\colon \mathbb{R} \to \mathbb{R}$ which shifts F into PC and which agrees with f on the set containing Q. In particular, g is an extendable function.

(iv) The last part of Theorem 9 is proved by the following inequality

$$\mathrm{A}(\mathrm{D}) = \mathrm{A}(\mathrm{AC}) = \mathrm{Add}(\mathrm{AC}, \mathrm{AC}) \leq \mathrm{Add}(\mathcal{F}_1, \mathcal{F}_2) \leq \mathrm{Add}(\mathrm{D}, \mathrm{D}) = \mathrm{A}(\mathrm{D}).$$

■

Proof of Theorem 10.

(i) To prove the first part of Theorem 10 we need one more lemma.

Lemma 27 $\mathrm{Add}(\mathrm{AD}, \mathrm{D}) \geq \mathrm{A}(\mathrm{D}(\mathrm{P}))$. *In particular,* $\mathrm{Add}(\mathrm{AD}, \mathrm{D}) = \mathrm{A}(\mathrm{D})$.

Proof. Let $P \subseteq \mathbb{R}$ be a perfect set with the property that $P \cup \{1\}$ is linearly independent over $\mathbb{Q}$. Observe that for every $p, q \in \mathbb{Q}$, $p \notin \{0, 1\}$ we have $(pP + q) \cap P = \emptyset$. Now, consider a countable partition $\{P_n \colon n < \omega\}$ of P into perfect sets. Using this partition and the above observation we can easily construct a family $\{P_n^\star \colon n < \omega\}$ of disjoint perfect sets such that $\bigcup_{n<\omega} P_n^\star$ is independent over $\mathbb{Q}$ and for every nontrivial interval $I \subseteq \mathbb{R}$ there is an $m < \omega$ such that $P_m^\star \subseteq I$. Note that $\bigcup_{n<\omega} P_n^\star$ is a $\mathfrak{c}$-dense meager F_σ-set.

To prove the inequality $\mathrm{Add}(\mathrm{AD}, \mathrm{D}) \geq \mathrm{A}(\mathrm{D}(\mathrm{P}))$ let us fix a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ such that $|F| < \mathrm{A}(\mathrm{D}(\mathrm{P}))$. There exists a function $g \in \mathbb{R}^{\mathbb{R}}$ satisfying the property $g + F \subseteq \mathrm{D}(\mathrm{P})$. We claim that if $g^\star \colon \mathbb{R} \to \mathbb{R}$ is any additive extension of $g|\bigcup_{n<\omega} P_n^\star$ then $g^\star + F \subseteq \mathrm{D}$. More precisely, for every $f \in F$, $g^\star + f$ is strongly Darboux. To see this pick any $f \in F$, $y \in \mathbb{R}$, and any interval I. There exists $m < \omega$ such that $P_m^\star$ is contained in I. Furthermore, we can find $x \in P_m^\star \subseteq I$ for which $g^\star(x) + f(x) = g(x) + f(x) = y$. This shows that $g^\star + f$ is strongly Darboux.

The second statement in the lemma is proved by the obvious inequality $\mathrm{A}(\mathrm{D}) \geq \mathrm{Add}(\mathrm{AD}, \mathrm{D}) \geq \mathrm{A}(\mathrm{D}(\mathrm{P}))$ and Proposition 26. ■

Now, (i) follows from Lemmas 24, 27, and Proposition 1 (1).

(ii) Since $\mathrm{Add}(\mathrm{AD}, \mathrm{Ext}) \leq \mathrm{A}(\mathrm{Ext}) = \mathfrak{c}^+$, it suffices to show the inequality $\mathrm{Add}(\mathrm{AD}, \mathrm{Ext}) \geq \mathfrak{c}^+$. So for every $F = \{f_\xi \colon \xi < \mathfrak{c}\} \subseteq \mathbb{R}^{\mathbb{R}}$ we need to find a $g \in \mathrm{AD}$ such that $g + F \subseteq \mathrm{Ext}$.

Let $\langle D_\xi \colon \xi < \mathfrak{c}\rangle$ be a sequence of pairwise disjoint $\mathfrak{c}-$dense meager F_σ sets such that $\bigcup_{\xi<\mathfrak{c}} D_\xi$ is linearly independent over $\mathbb{Q}$. Such a sequence can be constructed in a similar way as the $\mathfrak{c}-$dense meager F_σ-set in the proof of Lemma 27. Now, by [3, Proposition 4.3], for every $\xi < \mathfrak{c}$ we can find $h_\xi \in \mathrm{Ext}$ such that $\mathbb{R} \setminus D_\xi$ is h_ξ-negligible. We define g as an additive extension of $\bigcup_{\xi<\mathfrak{c}}(h_\xi - f_\xi)|D_\xi$.

To see that $g + f_\xi \in \mathrm{Ext}$ for every ξ, observe that $g + f_\xi = h_\xi$ on D_ξ. But the set $\mathbb{R} \setminus D_\xi$ is h_ξ-negligible. So each $g + f_\xi$ is extendable.

(iii) The prove of this part is similar to the prove of Theorem 2 (4). Fix a Hamel basis H which is a Bernstein set. By choosing the sets $B_{\langle I,p,m\rangle}$ to be subsets of H, we can obtain, for a given family F of real functions with cardinality less than $2^{\mathfrak{c}}$, an additive function which shifts F into PC.

(iv) Let us fix a family $F = \{h_\xi : \xi < \mathfrak{c}\} \subseteq \mathbb{R}^\mathbb{R}$ and a Hamel basis $H = \{x_\xi : \xi < \mathfrak{c}\}$. We will construct an additive function g with the property that $g + F \subseteq \mathrm{SZ}$, by defining it on H using induction. For a given $\alpha < \mathfrak{c}$, we choose

$$g(x_\alpha) \notin \left(\bigcup_{q\in\mathbb{Q}} \bigcup_{\xi,\gamma<\alpha} q(f_\gamma - h_\xi)[\mathrm{Lin}_\mathbb{Q}(x_\beta : \beta \le \alpha)] \right) + g[\mathrm{Lin}_\mathbb{Q}(x_\beta : \beta < \alpha)],$$

where $\langle f_\alpha : \alpha < \mathfrak{c}\rangle$ is a sequence of all continuous functions defined on G_δ subsets of $\mathbb{R}$. Such a choice is possible because the cardinality of the considered set is less than $\mathfrak{c}$. This choice also assures that $g + F \subseteq \mathrm{SZ}$. To see that observe the following $[g + h_\xi = f_\alpha] = [g = f_\alpha - h_\xi] \subseteq \mathrm{Lin}_\mathbb{Q}(x_\beta : \beta < \alpha)$ for all $\alpha, \xi < \mathfrak{c}$. Thus $|[g + h_\xi = f_\alpha]| = \omega\alpha < \mathfrak{c}$, which proves that $g + h_\xi \in \mathrm{SZ}$.

(v) First observe that $\mathrm{A}(\mathrm{AD}) = 2$. This follows from Proposition 1 (3)&(5) and obvious equality $\mathrm{AD} - \mathrm{AD} = \mathrm{AD}$. Recall also that $\mathrm{Add}(\mathcal{F}, \mathrm{AD}) \le \mathrm{A}(\mathrm{AD})$ and $\mathcal{F} - \mathrm{AD} = \mathrm{AD} - \mathcal{F} = \mathcal{F} + \mathrm{AD}$ for all $\mathcal{F} \in \{\mathrm{Ext}, \mathrm{AC}, \mathrm{Conn}, \mathrm{D}, \mathrm{PC}, \mathrm{SZ}\}$. Thus, by Proposition 1 (3) and Theorem 10 (i)-(iv), we get that $\mathcal{F} + \mathrm{AD} = \mathbb{R}^\mathbb{R}$. Consequently, $\mathrm{Add}(\mathcal{F}, \mathrm{AD}) = 2$.

The same part of Proposition 1 implies the second statement in (v). This is so because $\mathrm{C} - \mathrm{AD} = \mathrm{AD} - \mathrm{C} \neq \mathbb{R}^\mathbb{R}$. The characteristic function of a point, say $\chi_{\{0\}}$, is an example of a function witnessing the above property. Indeed, $(\chi_{\{0\}} + \mathrm{C}) \cap \mathrm{AD} = \emptyset$ because every additive function is either continuous or has a dense graph (see [2, Exercise 4, Section 7.3].) ■

References

[1] M. Balcerzak, K. Ciesielski, T. Natkaniec, *Sierpiński-Zygmund functions that are Darboux, almost continuous, or have a perfect road*, Arch. Math. Logic **37** (1997), 29–35.

[2] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Student Texts **39**, Cambridge Univ. Press 1997.

[3] K. Ciesielski, J. Jastrzębski, *Darboux-like functions within the classes of Baire one, Baire two, and additive functions*, Topology Appl. **103** (2000), 203–219.

[4] K. Ciesielski, A.W. Miller *Cardinal invariants concerning functions whose sum is almost continuous*, Real Anal. Exchange **20** (1994-95), 657-672.

[5] K. Ciesielski, T. Natkaniec, *Algebraic properties of the class of Sierpiński-Zygmund functions*, Topology Appl. **79** (1997), 75–99.

[6] K. Ciesielski, I. Recław, *Cardinal invariants concerning extendable and peripherally continuous functions*, Real Anal. Exchange **21** (1995-96), 459–472.

[7] R.G. Gibson, T. Natkaniec, *Darboux-like functions*, Real Anal. Exchange **22 (2)** (1996-97), 492–533.

[8] F. Jordan *Sums of Darboux-like functions from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$, Real Anal. Exchange **24 (2)** (1998-99), 729-759.

[9] A. Kamburelis, *On cardinal numbers related to Baire property*, Ph.D. Dissertation, Wrocław 1989.

[10] K.R. Kellum, *Sums and limits of almost continuous functions*, Colloq. Math. **31** (1974), 125–128.

[11] K.R. Kellum, *Almost Continuity and connectivity - sometimes it's as easy as to prove a stronger result*, Real Anal. Exchange **8** (1982-83), 244–252.

[12] K. Kuratowski, *Topologie I*, Warszawa 1958.

[13] T. Natkaniec, *Almost Continuity*, Real Anal. Exchange **17** (1991-92), 462–520.

[14] J.C. Oxtoby, *Measure and Category*, Springer-Verlag, Amsterdam 1971.