ON LINEABILITY OF ADDITIVE SURJECTIVE FUNCTIONS

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Abstract. We prove that the class of additive perfectly everywhere surjective functions contains (with the exception of the zero function) a vector space of maximal possible dimension ($2^{\mathfrak{c}}$). Additionally, we show under the assumption of regularity of $\mathfrak{c}$ that the family of additive everywhere surjective functions that are not strongly everywhere surjective contains (with the exception of the zero function) a vector space of dimension $\mathfrak{c}^+$.

The symbols $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of positive integers, rationa,l and real numbers, respectively. The cardinality of a set X is denoted by the symbol $|X|$. In particular, $|\mathbb{N}|$ is denoted by ω and $|\mathbb{R}|$ is denoted by $\mathfrak{c}$. We consider only real-valued functions. No distinction is made between a function and its graph. For any two partial real functions f, g we write $f+g$, $f-g$ for the sum and difference functions defined on $\text{dom}(f) \cap \text{dom}(g)$. We write $f|A$ for the restriction of f to the set $A \subseteq \mathbb{R}$. For any subset Y of a vector space V, any $v \in V$, and any $e \in \mathbb{R}$ we define $v + Y = \{v + y \colon y \in Y\}$ and $eY = \{ey \colon y \in Y\}$. Given $X \subseteq \mathbb{R}^n$, by $\text{span}_{\mathbb{Q}}\{\text{X}\}$ we denote the linear subspace of $\mathbb{R}^n$ over $\mathbb{Q}$ generated by X. It can be easily seen that $|\text{span}_{\mathbb{Q}}(\text{X})| = \max\{\omega, |\text{X}|\}$. Recall that a subset B of the real numbers is called a Bernstein set if $B \cap P \neq \emptyset$ and $\mathbb{R} \setminus B \cap P \neq \emptyset$ for every perfect set P. Bernstein sets are non-measurable and don't have the Baire property.

Recently, there have been lots of attention devoted to finding "large" structures (e.g., vector spaces, algebras) contained in various families of real functions (often with pathological properties) (see [1–5, 7–12, 15–17]). In this article we will concentrate on functions with various degree of surjectivity that are linear over the rationals (i.e., additive functions). An example of a function that is *everywhere surjective* (i.e., maps every non-empty open interval onto $\mathbb{R}$) was constructed by Lebesgue in [14]. Obviously, these functions are very discontinuous as their graph is a dense subset of $\mathbb{R}^2$. However, they are quite "common"

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among real functions. It turned out that the class contains a vector space (with the exception of the zero function) of the maximal possible dimension (see [3]). Later it was proved that the class of everywhere surjective additive functions also contains a vector space of the same dimension (see [11]). We will investigate additive functions that are surjective even in a stronger sense. The following are the definitions of functions considered in this paper.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is:

- *additive* ($f \in \mathrm{AD}$), if $f(x+y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$;
- *surjective* ($f \in \mathrm{SR}$), if $f(\mathbb{R}) = \mathbb{R}$;
- *everywhere surjective* ($f \in \mathrm{ES}$), if $f(I) = \mathbb{R}$ for every nonempty open interval I;
- *strongly everywhere surjective* ($f \in \mathrm{SES}$), if $|f^{-1}(\{y\}) \cap I| = \mathfrak{c}$ for every nonempty open interval I and every $y \in \mathbb{R}$;
- *perfectly everywhere surjective* ($f \in \mathrm{PES}$), if $f(P) = \mathbb{R}$ for every perfect set P.

Recall here, that additive functions are linear over $\mathbb{Q}$. In addition, if $X \subseteq \mathbb{R}$ is a set linearly independent over $\mathbb{Q}$ and $f\colon X \to \mathbb{R}$, then f can be extended to an additive function on $\mathbb{R}$. Observe also that $\mathrm{span}_{\mathbb{Q}}\{\mathrm{f}\}$ is a function whose domain is $\mathrm{span}_{\mathbb{Q}}\{\mathrm{X}\}$. If X is a Hamel basis (a basis for $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$), then $\mathrm{span}_{\mathbb{Q}}\{\mathrm{f}\}$ is the unique extension of f to an additive function on $\mathbb{R}$. Additionally, for any additive function f and $y \in \mathrm{range(f)}$ we have $f^{-1}(y) = x + \ker(f)$ for any $x \in f^{-1}(y)$ and therefore, $|f^{-1}(y)| = |\ker(f)|$. An additive function is injective if and only if $\ker(f) = \{0\}$ and if $\ker(f) \neq \{0\}$ then $\ker(f)$ is dense in $\mathbb{R}$ as it is a subspace of $\mathbb{R}$ over $\mathbb{Q}$ with the dimension at least 1.

We will recall now some definitions related to the theory of lineability (see [3, 5, 7]). Let V be a vector space over $\mathbb{R}$, $\mathcal{F} \subseteq V$, and κ be a cardinal number. We say $\mathcal{F}$ is *κ-lineable* if $\mathcal{F} \cup \{0\}$ contains a subspace of V of dimension κ. The *(coefficient of) lineability* of the subset $\mathcal{F}$ is denoted by $\mathcal{L}(\mathcal{F})$ and defined as follows

$$\mathcal{L}(\mathcal{F}) = \min\{\kappa\colon \mathcal{F} \text{ is not } \kappa\text{-lineable}\}.$$

It can be easily seen that AD is a subspace of $\mathbb{R}^{\mathbb{R}}$. Since additive functions can be arbitrarily defined on a Hamel basis we can also conclude that $|\mathrm{AD}| = 2^{\mathfrak{c}}$. Hence $\mathcal{L}(\mathrm{AD}) = (2^{\mathfrak{c}})^{+}$. The families of functions with various degree of surjectivity have also been studied in the context of lineability. All of these families contain (with the exception of the zero function) a vector space of maximal possible dimension ($2^{\mathfrak{c}}$). Specifically, in [10], the authors show that $\mathcal{L}(\mathrm{PES}) = \mathcal{L}(\mathrm{SES} \setminus \mathrm{PES}) = (2^{\mathfrak{c}})^+$. We will show that these results hold within the class of additive functions. Additionally, in [11] an analogous result is proved for the additive everywhere surjective functions $\mathrm{AD} \cap \mathrm{ES}$ and additive functions that are not surjective $\mathrm{Add} \setminus \mathrm{SR}$ (actually the latter is stated for $\mathrm{Add} \setminus \mathrm{ES}$ but the proof shows that $\mathcal{L}(\mathrm{Add} \setminus \mathrm{SR}) = (2^{\mathfrak{c}})^+$). In this note, we prove the following results.

Theorem 1. $\mathcal{L}(\mathrm{AD} \cap \mathrm{PES}) = \mathcal{L}(\mathrm{AD} \cap \mathrm{SES}) = \mathcal{L}(\mathrm{AD} \cap \mathrm{SES} \setminus \mathrm{PES}) = (2^{\mathfrak{c}})^+$.

Theorem 2. *Suppose that* $\mathrm{cof}(\mathfrak{c}) = \mathfrak{c}$. *Then* $\mathcal{L}(\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}) > \mathfrak{c}^+$. *Hence, it is consistent with ZFC (e.g., under the assumption of GCH) that* $\mathcal{L}(\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}) = \mathcal{L}(\mathrm{ES} \setminus \mathrm{SES}) = (2^{\mathfrak{c}})^+$.

Before proceeding to the proofs of the above results let us make an observation about the lineability of the class of additive surjective functions that are not everywhere surjective.

Remark 3. $\mathcal{L}(\mathrm{AD} \cap \mathrm{SR} \setminus \mathrm{ES}) = 2$.

The above remark easily follows from [11, Theorem 4.1], which states that for additive functions, being everywhere surjective is equivalent to being surjectve and not injective. Hence the class $\mathrm{AD} \cap \mathrm{SR} \setminus \mathrm{ES}$ contains only injective functions and it is known that the lineability of the class of one-to-one functions is equal to 2 (see [17]).

Proof of Theorem 1. Let $V \subseteq \mathbb{R}^{\mathbb{R}}$ be a subspace such that $|V| = 2^{\mathfrak{c}}$ and $V \setminus \{0\} \subseteq \mathrm{SR}$ (see [3]). Let B be a Bernstein set that is a Hamel basis. It is well know that it can be represented as the union of two disjoint Bernstein sets (actually, every Bernstein set can be decomposed into $\mathfrak{c}$ many Bernstein sets - see [6]). Hence let $B = B_1 \cup B_2$, where B_1 and B_2 are two dosjoint Bernstein sets. Fix a bijection $g \colon B_1 \to \mathbb{R}$ and choose $h \in V$. Define an additive function $f_h \colon \mathbb{R} \to \mathbb{R}$ as a unique function such that $f_h|B_1 = h \circ g|B_1$ and $f_h(B_2) = \{0\}$. Then $W = \{f_h \colon h \in V\}$ is a subspace of $(\mathrm{AD} \cap \mathrm{PES}) \cup \{0\}$ of dimension $2^{\mathfrak{c}}$. Indeed, for every $f_h \in W \setminus \{0\}$, $B_2 \subseteq \ker(f_h)$. Hence, since f_h is surjective we get that $f_h^{-1}(y)$ is a Bernstein set for every $y \in \mathbb{R}$ (as it is a translation of $\ker(f_h)$). This implies that $f_h(P) = \mathbb{R}$ for every perfect set P and we can conclude that $f_h \in \mathrm{PES}$. Since $\mathrm{PES} \subseteq \mathrm{SES}$, this also clearly shows that $\mathcal{L}(\mathrm{AD} \cap \mathrm{PES}) = \mathcal{L}(\mathrm{AD} \cap \mathrm{SES}) = (2^{\mathfrak{c}})^+$.

Regarding the equality $\mathcal{L}(\mathrm{AD} \cap \mathrm{SES} \setminus \mathrm{PES}) = (2^{\mathfrak{c}})^+$, first let us note that our argument is similar to the one used in the proof of [10, Theorem 2.7]. For a reader's convenience we include an outline of the proof. Since our goal is to construct a vector subspace of $\mathrm{SES} \setminus \mathrm{PES}$ consisting of additive functions, we will use a perfect set $P \subseteq \mathbb{R}$ that is linearly independent over $\mathbb{Q}$. Construction of such a set can be found in [13]. Next, decompose P into two perfect sets, that is let $P = P_1 \cup P_2$, where P_1 and P_2 are perfect sets such that $P_1 \cap P_2 = \emptyset$. Now, fix a bijection $g\colon P_1 \to \mathbb{R}$ and choose $h \in V$. Define an additive function $f_h\colon \mathbb{R} \to \mathbb{R}$ as a unique function such that $f_h|P_1 = h \circ g|P_1$ and $f_h(X \setminus P_1) = \{0\}$, where X is a Hamel basis containing P (obviously, $P_2 \subseteq X$, and therefore $f_h(P_2) = \{0\}$). Then $W = \{f_h \colon h \in V\}$ is a subspace of $(\mathrm{AD} \cap \mathrm{SES} \setminus \mathrm{PES}) \cup \{0\}$ of dimension $2^{\mathfrak{c}}$. □

Proof of Theorem 2. Let $\mathcal{F} = \{f_\gamma : \gamma < \mathfrak{c}\} \subseteq (\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}) \cup \{0\}$ be a vector space of dimension $\leq \mathfrak{c}$. We will show that there exists an $h \in (\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}) \setminus \mathcal{F}$ such that $h + \mathcal{F} \subseteq \mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}$. Since $\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}$ is closed under scalar multiplication (of course except for 0) the latter will imply that $\{ah \colon a \in \mathbb{R}\} + \mathcal{F}$ is a vector space in $(\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}) \cup \{0\}$ such that $\mathcal{F} \subsetneq \{ah \colon \in \mathbb{R}\} + \mathcal{F}$. Using Zorn's lemma, we can then conclude that $(\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}) \cup \{0\}$ contains a vector space of dimension $\mathfrak{c}^+$.

Let $\mathcal{G} = \{g_\alpha : \alpha < \mathfrak{c}\}$ ($g_0 \equiv 0$) be the set of all constant functions on $\mathbb{R}$ (without repetitions) and $X = \{x_\alpha : \alpha < \mathfrak{c}\}$ be a Hamel basis for $\mathbb{R}$. We will construct by induction a sequence of partial functions h_α ($\alpha < \mathfrak{c}$) such that:

(i) $h_\xi \subseteq h_\alpha$ for $\xi < \alpha$;

(ii) $|\mathrm{dom}(h_\alpha)| \leq \max(\omega, \alpha)$, $x_\alpha \in \mathrm{dom}(h_\alpha) \subseteq X$, and $h_\alpha \cap f_\alpha \neq h_\alpha$;

(iii) $(g_\zeta \cap (f_\gamma + \mathrm{span}_{\mathbb{Q}}\{\mathrm{h}_\alpha\}) \subseteq (\mathrm{f}_\gamma + \mathrm{span}_{\mathbb{Q}}\{\mathrm{h}_\xi\})$ for $\zeta, \gamma \leq \xi < \alpha$;

(iv) $\big(g_\zeta \cap (f_\gamma + \mathrm{span}_{\mathbb{Q}}\{\mathrm{h}_\alpha\})\big) \setminus \{(0,0)\} \neq \emptyset$ for $\zeta, \gamma \leq \alpha$.

We start the construction by defining $h_0(x_0) = (g_0 - f_0)(x_0)$ and $h_0(x_1) \neq f_0(x_1)$. It is easy to see that h_0 satisfies all the conditions (i)-(iv).

Now fix $\alpha < \mathfrak{c}$ and assume that the sequence h_β has been defined for all $\beta < \alpha$ satisfying the conditions (i)-(iv). Put $h_\alpha = \bigcup_{\beta<\alpha} h_\beta$. If $x_\alpha \notin \mathrm{dom}(h_\alpha)$, then let $x = x_\alpha$. Otherwise, let x be any element of $X \setminus \mathrm{dom}(h_\alpha)$. Next choose

$$h_\alpha(x) \in \mathbb{R} \setminus \mathrm{span}_{\mathbb{Q}} \left\{ \bigcup_{\gamma \leq \alpha, \beta < \alpha} \mathrm{range(f_\gamma + span_{\mathbb{Q}}\{h_\beta\}) \cup \{f_\gamma(x)\}} \right\}.$$

This choice is possible since $|\mathrm{dom(span_{\mathbb{Q}}\{h_\beta\})}| < \mathfrak{c}$ for all $\beta < \alpha$ and we are assuming that $\mathfrak{c}$ is regular. In addition, this choice ensures that the condition (iii) is being preserved. Indeed, let $\xi < \alpha$ and $\zeta, \gamma \leq \xi$. Since the condition (iv) is satisfied for all $\beta < \alpha$ and $\bigcup h_\beta \subseteq h_\alpha$ we have that $\big(g_\zeta \cap (f_\gamma + \mathrm{span}_{\mathbb{Q}}\{h_\alpha\})\big) \neq \emptyset$. Hence, to see that

$$(g_\zeta \cap (f_\gamma + \mathrm{span}_{\mathbb{Q}}\{h_\alpha\}) \subseteq (\mathrm{f_\gamma + span_{\mathbb{Q}}\{h_\xi\}})$$

it is sufficient to show that $(\mathrm{span_{\mathbb{Q}}\{h_\alpha\} + f_\gamma)(rx + t) \notin range(f_\gamma + span_{\mathbb{Q}}\{\bigcup h_\beta\}})$ for all $r \in \mathbb{Q} \setminus \{0\}$ and $t \in \mathrm{span}_{\mathbb{Q}}\{\mathrm{dom}(\bigcup \mathrm{h}_\beta)\}$. Assume to the contrary. This would imply that $(h_\alpha + f_\gamma)(x) \in \mathrm{range(f_\gamma + span_{\mathbb{Q}}\{h_\beta\})}$ for some $\beta < \alpha$, which would contradict the choice of $h_\alpha(x)$.

Next notice that, since the conditions (i)-(iv) are satisfied for $\beta < \alpha$, we can conclude that for $\zeta, \gamma < \alpha$ we have

$$\big(g_\zeta \cap (f_\gamma + \mathrm{span}_{\mathbb{Q}}\{h_\alpha\})\big) \setminus \{(0,0)\} \neq \emptyset.$$

In the next step of the construction we will ensure that the above condition is also satisfied for $\zeta = \alpha$ or $\gamma = \alpha$.

Let $\zeta = \alpha$ and $\gamma = 0$. Suppose that $g_\alpha \cap (f_0 + \mathrm{span}_{\mathbb{Q}}\{h_\alpha\}) \setminus \{(0,0)\} = \emptyset$. Choose

$$x \in X \setminus \mathrm{span}_{\mathbb{Q}} \left\{ \mathrm{dom(h_\alpha)} \cup \bigcup_{\kappa < \alpha, \kappa \neq 0} (\mathrm{f}_\kappa - \mathrm{f}_0)^{-1} \left(\mathrm{span}_{\mathbb{Q}}\{\mathrm{range(f_\kappa + h_\alpha) \cup \{g_\alpha(0)\}}\}\right) \right\}$$

and define $h_\alpha(x) = g_\alpha(x) - f_0(x)$. The above choice is possible since we are assuming the regularity of $\mathfrak{c}$ and $f_\kappa - f_0 \in \mathrm{Add} \cap \mathrm{ES} \setminus \mathrm{SES}$ for $\kappa \neq 0$, which implies that $|(f_\kappa - f_0)^{-1}(y)| = |\ker(f_\kappa - f_0)| < \mathfrak{c}$ for every $y \in \mathbb{R}$.

Obviously, $\big(g_\alpha \cap (f_0 + \mathrm{span}_{\mathbb{Q}}\{h_\alpha\})\big) \setminus \{(0,0)\} \neq \emptyset$. Next we will verify that the condition (iii) still holds. Let $\xi < \alpha$ and $\tau, \kappa \leq \xi$. We have that $g_\tau \cap (f_\kappa + \mathrm{span_{\mathbb{Q}}\{h_\alpha \setminus \{(x, h_\alpha(x))\}\}}) \neq \emptyset$ and

$$(g_\tau \cap (f_\kappa + \mathrm{span_{\mathbb{Q}}\{h_\alpha \setminus \{(x, h_\alpha(x))\}\}}) \subseteq (\mathrm{f_\kappa + span_{\mathbb{Q}}\{h_\xi\}}).$$

Assume that $(\mathrm{span_{\mathbb{Q}}\{h_\alpha\} + f_\kappa)(rx + t) \in range(f_\kappa + span_{\mathbb{Q}}\{h_\alpha \setminus \{(x, h_\alpha(x))\}\}})$ for some rational number $r \neq 0$ and $t \in \mathrm{span_{\mathbb{Q}}\{dom(h_\alpha) \setminus \{x\}\}}$. This would imply that $(h_\alpha + f_\kappa)(x) \in$

$\mathrm{span}_{\mathbb{Q}}\{\mathrm{range}(\mathrm{f}_\kappa + \mathrm{span}_{\mathbb{Q}}\{\mathrm{h}_\alpha \setminus \{(\mathrm{x}, \mathrm{h}_\alpha(\mathrm{x}))\}\})\} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{range}(\mathrm{f}_\kappa + \mathrm{h}_\alpha \setminus \{(\mathrm{x}, \mathrm{h}_\alpha(\mathrm{x}))\})\}$. If $\kappa = 0$, then we would conclude that $g_\alpha \cap (f_0 + \mathrm{span}_{\mathbb{Q}}\{\mathrm{h}_\alpha \setminus \{(\mathrm{x}, \mathrm{h}_\alpha(\mathrm{x}))\}\}) \setminus \{(0,0)\} \neq \emptyset$, which contradicts our assumption. If $\kappa \neq 0$, then we would obtain that

$$x \in (f_\kappa - f_0)^{-1}\left(\mathrm{span}_{\mathbb{Q}}\{\mathrm{range}(\mathrm{f}_\kappa + \mathrm{h}_\alpha \setminus \{(\mathrm{x}, \mathrm{h}_\alpha(\mathrm{x}))\}) \cup \{\mathrm{g}_\alpha(0)\}\}\right),$$

which contradicts the way x was selected.

It is not difficult to notice that the above process can be repeated through induction for $\gamma \leq \alpha$ resulting in an extended h_α having the property $g_\alpha \cap (f_\gamma + \mathrm{span}_{\mathbb{Q}}\{\mathrm{h}_\alpha\}) \setminus \{(0,0)\} \neq \emptyset$. Observe also that (by the above argument) the condition (iii) will be preserved in each step of the inductive process. Similarly, we can extend h_α so that $g_\zeta \cap (f_\alpha + \mathrm{span}_{\mathbb{Q}}\{\mathrm{h}_\alpha\}) \setminus \{(0,0)\} \neq \emptyset$ for $\zeta < \alpha$ and again, the condition (iii) still holds.

This completes the inductive definition of the sequence of functions h_α ($\alpha < \mathfrak{c}$) satisfying the conditions (i)-(iv). Notice that $\mathrm{dom}(\bigcup_{\alpha<\mathfrak{c}} h_\alpha) = X$. Define h to be the unique additive function such that $h|X = \bigcup_{\alpha<\mathfrak{c}} h_\alpha$, that is $h = \mathrm{span}_{\mathbb{Q}}\{\bigcup_{\alpha<\mathfrak{c}} \mathrm{h}_\alpha\}$. The condition (ii) implies that $h \notin \mathcal{F}$. Additionally, based on the condition (iv) we conclude that for every $\gamma < \mathfrak{c}$ we have $h + f_\gamma \in \mathrm{SR}$ and $\ker(\mathrm{h} + \mathrm{f}_\gamma) \neq \{0\}$. Hence $\ker(\mathrm{h} + \mathrm{f}_\gamma)$ is dense in $\mathbb{R}$ and consequently, $(h + f_\gamma)^{-1}(y)$ is also dense in $\mathbb{R}$ for every $y \in \mathbb{R}$. However, it follows from the condition (iii) that $|(h + f_\gamma)^{-1}(y)| < \mathfrak{c}$ for every $y \in \mathbb{R}$. Therefore $h + f_\gamma \in \mathrm{Add} \cap \mathrm{ES} \setminus \mathrm{SES}$. □

Let us make here some remarks regarding the last result. Given that in [7] it is proved in ZFC that $\mathcal{L}(\mathrm{ES} \setminus \mathrm{SES}) > \mathfrak{c}^+$ (Corollary 2.15), one may wonder if the inequality from Theorem 2 in this note could similarly be proved without the use of additional set-theoretic assumptions. However, recall that in the same article the authors use the assumption of regularity of $\mathfrak{c}$ to show that $\mathcal{L}(F_{<\mathfrak{c}} \cap \mathrm{SES}) > \mathfrak{c}^+$ (where $F_{<\mathfrak{c}}$ is a family of real functions for which preimages of singletons have cardinality less than $\mathfrak{c}$) and $\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}$ is a proper subclass of $F_{<\mathfrak{c}} \cap \mathrm{SES}$ (indeed, for every $f \in \mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}$ and $y \in \mathbb{R}$ we have that $f^{-1}\{y\}$ is dense and $|f^{-1}\{y\}| = |\ker f| < \mathfrak{c}$). The difficulty when constructing a "large" subspace of $\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}$ lies in the fact that one needs to carefully design the inductive process to control the cardinality of the kernels and the author was not able to eliminate the need for the assumption $\mathrm{cof}(\mathfrak{c}) = \mathfrak{c}$. Consequently, we state the following question.

Problem 4. Can it be proved in ZFC that $\mathcal{L}(\mathrm{AD} \cap \mathrm{ES} \setminus \mathrm{SES}) = \mathcal{L}(\mathrm{ES} \setminus \mathrm{SES})$?

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