Weak Subordination for Convex Univalent Harmonic Functions *†

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Abstract

For two complex-valued harmonic functions f and F defined in the open unit disk Δ with $f(0) = F(0) = 0$, we say f is weakly subordinate to F if $f(\Delta) \subset F(\Delta)$. Furthermore, if we let E be a possibly infinite interval, a function $f : \Delta \times E \to \mathbb{C}$ with $f(\cdot, t)$ harmonic in Δ and $f(0, t) = 0$ for each $t \in E$ is said to be a weak subordination chain if $f(\Delta, t_1) \subset f(\Delta, t_2)$ whenever $t_1, t_2 \in E$ and $t_1 < t_2$. In this paper, we construct a weak subordination chain of convex univalent harmonic functions using a harmonic de la Vallée Poussin mean and a modified form of Pommerenke's criterion for a subordination chain of analytic functions.

1 Introduction

For analytic functions f and g defined in the open unit disk Δ with $f(0) = g(0) = 0$, f is subordinate to g, written $f \prec g$, if there exists an analytic function $\phi : \Delta \to \mathbb{C}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$, $z \in \Delta$, such that $f(z) = g(\phi(z))$. A natural extension of subordination to complex-valued harmonic functions f and F in Δ with $f(0) = F(0) = 0$ is to say f is subordinate to F if $f(z) = F(\phi(z))$ where ϕ is analytic in Δ, $|\phi(z)| < 1$, $z \in \Delta$, and $\phi(0) = 0$. See [8] for results relating to this definition. There are a few limitations to this definition because ϕ must be analytic to preserve harmonicity and, even if $f(\Delta) \subset F(\Delta)$ and F is one-to-one, such a ϕ may not exist as is the case for analytic functions. If f and F are harmonic functions on Δ with $f(0) = F(0) = 0$, we say f is weakly subordinate to F if $f(\Delta) \subset F(\Delta)$. Furthermore, if we let E be a possibly infinite interval, a function $f : \Delta \times E \to \mathbb{C}$ with $f(\cdot, t)$ harmonic in Δ and $f(0, t) = 0$ for each $t \in E$ is said to be a weak subordination chain if $f(\Delta, t_1) \subset f(\Delta, t_2)$ whenever $t_1, t_2 \in E$ and $t_1 < t_2$. In this paper, we will construct a weak subordination chain of convex univalent harmonic functions.

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Every complex-valued harmonic function f in Δ with $f(0) = 0$ can be uniquely represented as $f = h + \overline{g}$ where h and g are analytic in Δ and $h(0) = g(0) = 0$. In addition, $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is sense-preserving if the Jacobian, J_f, of the mapping $(x, y) \mapsto (u, v)$ is positive. The function f is locally univalent if J_f never vanishes in Δ. By a result of Lewy [3] a harmonic mapping $f : \Delta \to \mathbb{C}$ of the form $f = h + \overline{g}$, is locally univalent and sense-preserving if, and only if, $|g'(z)| < |h'(z)|$ for all $z \in \Delta$. In this case, we simply say f is locally univalent. In addition, we say f is univalent if f is one-to-one and sense-preserving in Δ. Let $\mathcal{S}_H$ be the family of harmonic univalent functions in Δ of the form $f = h + \overline{g}$ with $h(0) = g(0) = 0$ and $h'(0) = 1$. Let $\mathcal{K}_H$ be the set of functions in $\mathcal{S}_H$ such that $f(\Delta)$ is convex. We will simply say f is convex if $f(\Delta)$ is convex.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in Δ. Then the function $f * g$ given by $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ is called the Hadamard product of f and g. Let $\mathcal{H}_0(\Delta)$ be the set of analytic functions in Δ with $f(0) = 0$. For $f \in \mathcal{H}_0$, $f * I = f$ where I is the half-plane mapping

$$I(z) = \frac{z}{1 - z}. \tag{1}$$

In [4], Pólya and Schoenberg studied the shape-preserving properties of the de la Vallée Poussin means. Define

$$V_n(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} z^k, \qquad n \in \mathbb{N}, z \in \Delta.$$

For $f \in \mathcal{H}_0(\Delta)$, $V_n * f$ is the n^{th} de la Vallée Poussin mean of f. In 2003, Ruscheweyh and Suffridge [7] proved V_n satisfies the differential equation

$$zV_\lambda'(z) + \lambda \frac{1 - z}{1 + z} V_\lambda(z) = \lambda \frac{z}{1 + z}, \quad V_\lambda(0) = 0 \tag{2}$$

when $\lambda = n$. Furthermore, the differential equation has analytic solutions for $\lambda > 0$, and the solutions form a continuous extension of the de la Vallée Poussin means. Let $\mathcal{K}$ denote the set of convex univalent functions in $\mathcal{H}_0(\Delta)$ with $f'(0) = 1$. Ruscheweyh and Suffridge [7] proved the following theorem involving a convex subordination chain resolving a conjecture of Pólya and Schoenberg posed in [4].

Theorem 1 (Ruscheweyh, Suffridge). *For $f \in \mathcal{K}$, we have $((\lambda+1)/\lambda)V_\lambda * f \in \mathcal{K}$, for all $\lambda > 0$. Furthermore,*

$$V_{\lambda_1} * f \prec V_{\lambda_2} * f \prec f, \qquad 0 < \lambda_1 < \lambda_2.$$

In particular, $V_{\lambda_1} \prec V_{\lambda_2} \prec I$, $0 < \lambda_1 < \lambda_2$ where I is given by equation (1) and in fact, this special case implies the truth of Theorem 1. See [6].

If $f \in \mathcal{H}_0(\Delta)$ and $F = H + \overline{G}$ is a harmonic function in Δ with $F(0) = 0$, then the Hadamard product or convolution of f and F is defined as

$$f \,\tilde{*}\, F = f * H + \overline{f * G}$$

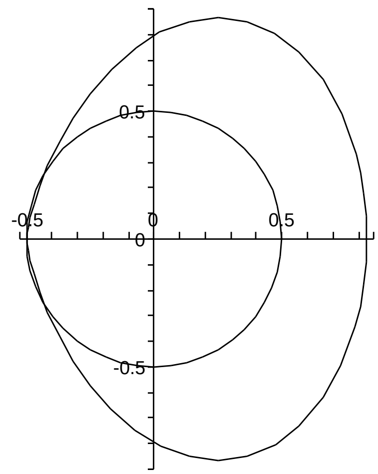

Figure 1: $V_1 \tilde{*} \ell_0(\Delta) \not\subset V_2 \tilde{*} \ell_0(\Delta)$

where $f*F$ and $f*G$ are the usual Hadamard products of two analytic functions. Then $V_\lambda \tilde{*} F$ is the harmonic de la Vallée Poussin mean of F. We have the following result of Ruscheweyh and Suffridge [7] regarding harmonic de la Vallée Poussin means and their shape-preserving property.

Theorem 2 (Ruscheweyh, Suffridge)**.** *For $\lambda \geq 1/2$, if F is a convex univalent harmonic function in Δ, then so is $V_\lambda \tilde{*} F$, and $V_\lambda \tilde{*} F(\Delta) \subset F(\Delta)$.*

The half-plane mapping

$$\begin{aligned} \ell_0(z) &= \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right) + \frac{1}{2}\overline{\left(\frac{z}{1-z} - \frac{z}{(1-z)^2}\right)} \\ &= \frac{I(z) + zI'(z)}{2} + \frac{\overline{I(z) - zI'(z)}}{2} \end{aligned}$$

is convex univalent and harmonic in Δ (see [1]) and is the harmonic analogue to the analytic half-plane mapping I given by (1). One might hope that at least the mapping $(z, \lambda) \mapsto V_\lambda \tilde{*} \ell_0$ would form a weak subordination chain paralleling the analytic case. Unfortunately $V_1 \tilde{*} \ell_0(\Delta) \not\subset V_2 \tilde{*} \ell_0(\Delta)$ (see [7]) which is illustrated in Figure 1 and Figure 2. Therefore, even a weak subordination chain result as in Theorem 1 does not hold for every convex harmonic function. However, by adjusting $V_\lambda \tilde{*} \ell_0$ we construct a weak subordination chain of convex univalent harmonic functions.

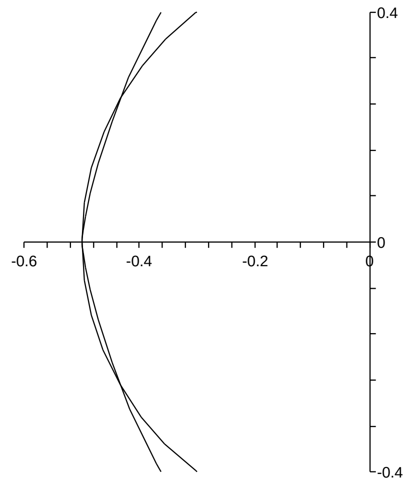

Figure 2: Above is a partial graph of $V_1 \tilde{*} \ell_0(e^{i\theta})$ and $V_2 \tilde{*} \ell_0(e^{i\theta})$.

2 A Weak Subordination Chain of Convex Univalent Harmonic Functions

Define $F : \Delta \times [1/2, \infty) \to \mathbb{C}$ by

$$F(z, \lambda) = \frac{V_\lambda(z) + c_\lambda z V_\lambda'(z)}{1 + c_\lambda} + \overline{\frac{V_\lambda(z) - c_\lambda z V_\lambda'(z)}{1 + c_\lambda}} \tag{3}$$

where $c_\lambda > 0$ for each $\lambda \in [1/2, \infty)$. Observe that if $c_\lambda = 1$ for all $\lambda \geq 1/2$, then $F(z, \lambda) = (V_\lambda \tilde{*} \ell_0)(z)$. If we let c_λ become unbounded in equation (3), $V_\lambda(z)/(1 + c_\lambda)$ is approaching the zero function while $(c_\lambda z V_\lambda'(z))/(1 + c_\lambda)$ is approaching $z V_\lambda'(z)$, a starlike function. Surprisingly, however, the functions $F(\cdot, \lambda)$ form a family of convex univalent harmonic functions for $c_\lambda > 0$, which is formally stated below.

Theorem 3. *For each $\lambda \geq 1/2$ and $c_\lambda > 0$, $((\lambda + 1)/\lambda) F(\cdot, \lambda) \in \mathcal{K}_H$.*

In the proof of Theorem 3, which is given in the next section, we use the fact that the functions $F(\cdot, \lambda)$ can be realized as harmonic de la Vallée Poussin means of a convex harmonic function. That is, the function F can be written as

$$F(z, \lambda) = (V_\lambda \tilde{*} I_\lambda)(z)$$

where

$$I_\lambda(z) = \frac{I(z) + c_\lambda z I'(z)}{1 + c_\lambda} + \overline{\frac{I(z) - c_\lambda z I'(z)}{1 + c_\lambda}}, \qquad z \in \Delta, \lambda \geq 1/2, c_\lambda > 0 \tag{4}$$

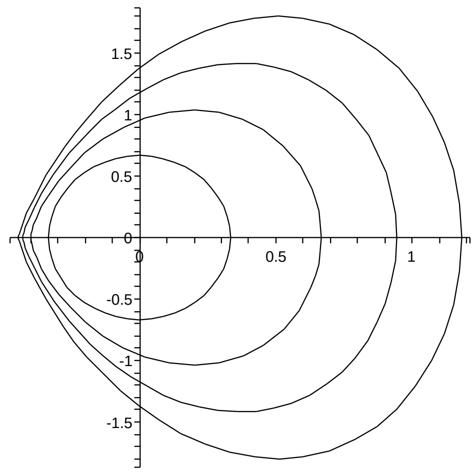

Figure 3: Let $c_\lambda = 1 + 1/\lambda$. Above are the graphs of $F(e^{i\theta}, \lambda)$, $\lambda = 1, 2, 3, 4$.

and I is given by (1). In the proof of Theorem 3, it is shown that I_λ is a convex harmonic half-plane mapping, and therefore, by Theorem 2, we have the following corollary.

Corollary 4. *For each $\lambda \geq 1/2$ and $c_\lambda > 0$, $F(\Delta, \lambda) \subset I_\lambda(\Delta)$.*

Next, for $\lambda \geq 1$ and a specific choice of c_λ in (3), we construct a convex univalent weak subordination chain. See Figure 3 for an illustration.

Theorem 5. *If $c_\lambda = 1 + 1/\lambda$ and $\lambda \geq 1$, F is a convex univalent weak subordination chain.*

Notice for $c_\lambda = 1 + 1/\lambda$, F tends to ℓ_0 as $\lambda \to \infty$. As can be seen in the next section, the proof of Theorem 5 is complicated by the involvement of the Gamma and Psi functions. We believe in fact that $F(\Delta, \lambda_1) \subset F(\Delta, \lambda_2)$ for $1/2 \leq \lambda_1 < \lambda_2$ when $c_\lambda = 1+1/\lambda$. Furthermore, whether there are other choices of c_λ for which F is a weak subordination chain remains an open question.

3 Proofs

Proof of Theorem 3. Let $\lambda \geq 1/2$ be fixed and $c_\lambda > 0$. Recall $F(z, \lambda) = (V_\lambda \tilde{*} I_\lambda)(z)$ where F is given by equation (3) and I_λ is given by equation (4). By Theorem 2, to show $F(\cdot, \lambda)$ is convex, it suffices to show that $I_\lambda \in \mathcal{K}_H$. To do this, we will perform a change of variable. Substituting $u = (1 + z)/(1 - z)$

into I_λ, we can study the image $I_\lambda(z)$ as z varies in Δ by studying $g(u)$ as $u = x + iy$ varies in the right-half plane where

$$\operatorname{Re} g(u) = \frac{1}{1+c_\lambda} \operatorname{Re}(u-1) = \frac{1}{1+c_\lambda}(x-1)$$

and

$$\operatorname{Im} g(u) = \frac{2}{1+c_\lambda} \operatorname{Im}\left(\frac{1}{4}(u^2-1)\right) = \frac{1}{1+c_\lambda} xy.$$

Setting $x = 0$, we see that g takes the unit circle except $z = 1$ to the point $(-1/(1+c_\lambda), 0)$. Setting $x = k > 0$, we see that for the fixed real value of $(k-1)/(1+c_\lambda)$, g will will take on all imaginary values. Thus, g maps the open right-half plane into the half-plane $\{w : \operatorname{Re} w > -1/(1+c_\lambda)\}$. To see that I_λ is one-to-one, suppose there exist $u_1 = x_1 + iy_1$ and $u_2 = x_2 + iy_2$ with $x_1, x_2 > 0$ such that $g(u_1) = g(u_2)$. By the above work, this implies $x_1 = x_2$ which in turn implies $y_1 = y_2$. Since I_λ is one-to-one, it is either sense-preserving or sense-reversing on the entire disk Δ. Write $I_\lambda = H_\lambda + \overline{G_\lambda}$. Then $H_\lambda'(0) = I'(0) = 1$ and $G_\lambda'(0) = (1-c_\lambda)/(1+c_\lambda)I'(0) = (1-c_\lambda)/(1+c_\lambda)$. Consequently, $|G_\lambda'(0)| < |H_\lambda'(0)|$ when $c_\lambda > 0$, and I_λ is sense-preserving. Finally, since $H_\lambda(0) = G_\lambda(0) = 0$ and $H_\lambda'(0) = 1$, $I_\lambda \in \mathcal{K}_H$. □

To prove Theorem 5, we require a modified form of Pommerenke's criterion [5] for a subordination chain of analytic functions to apply to a weak subordination chain of harmonic functions.

Theorem 6. *Let $a < b$ and $f : \overline{\Delta} \times [a,b] \to \mathbb{C}$. Suppose $f(\cdot, t)$ is harmonic on $\overline{\Delta}$, univalent in Δ, and $f(0,t) = 0$ for each $t \in [a,b]$. Further, assume $f(z,\cdot) \in \mathcal{C}^1[a,b]$ for each $z \in \Delta$. Write $f(z,t) = h(z,t) + \overline{g(z,t)}$. If $p(z,t)$ given by*

$$\frac{\partial f(z,t)}{\partial t} = p(z,t)\left(z\frac{\partial h(z,t)}{\partial z} - \overline{z\frac{\partial g(z,t)}{\partial z}}\right), \quad |z| = 1, t \in [a,b]$$

has $\operatorname{Re} p(z,t) > 0$, $|z| = 1$, $t \in [a,b]$, then f is a weak subordination chain.

Proof. For z fixed, $|z| = 1$, we can think of $f(z,t)$ as the path of a particle. The vector given by $[\partial f/\partial t](z,t)$ represents the velocity; while,

$$z\frac{\partial h(z,t)}{\partial z} - \overline{z\frac{\partial g(z,t)}{\partial z}}$$

for $|z| = 1$ is the normal. If $\operatorname{Re} p(z,t) > 0$ for $|z| = 1$ and each $t \in [a,b]$, then the velocity vector and the normal must be within $\pi/2$ of one another for every z, $|z| = 1$. This implies that the direction of the velocity vector at every boundary point of $\{f(z,t) : |z| \le 1\}$ is toward the exterior of the set. Let $s \in [a,b)$. Then for any ε, $0 < \varepsilon \le b - s$, $f(\Delta, s) \subset f(\Delta, s+\varepsilon)$, and hence, $f(\Delta, s) \subset f(\Delta, t)$ for all s and t such that $a \le s < t \le b$. □

The following lemmas are needed for the proof of Theorem 5.

Lemma 7. *For $\lambda \geq 1$,*

$$-\log 2 < \log \frac{\lambda+1}{2\lambda+1} + \frac{1}{2\lambda+1} - \frac{1}{\lambda+1} \leq \Psi(\lambda+1) - \Psi(2\lambda+1) \leq \log \frac{\lambda+1}{2\lambda+1}$$

where $\Psi(z)$ is the digamma function.

Proof. Since

$$\Psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right)$$

where γ is the Euler-Mascheroni constant,

$$\Psi(\lambda+1) - \Psi(2\lambda+1) = \sum_{n=0}^{\infty} \left(\frac{1}{2\lambda+1+n} - \frac{1}{\lambda+1+n} \right) = \sum_{n=0}^{\infty} f(n).$$

Then $f(t) = 1/(2\lambda+1+t) - 1/(\lambda+1+t)$ is an increasing function of t and is negative for $t \geq 0$ and $\lambda \geq 1$. Therefore, for $\lambda \geq 1$,

$$\Psi(\lambda+1) - \Psi(2\lambda+1) \leq \int_0^{\infty} f(t)\,dt = \log\left(\frac{\lambda+1}{2\lambda+1}\right).$$

Similarly, we conclude

$$\begin{aligned} \Psi(\lambda+1) - \Psi(2\lambda+1) &= \frac{1}{2\lambda+1} - \frac{1}{\lambda+1} + \sum_{n=1}^{\infty} f(n) \\ &\geq \frac{1}{2\lambda+1} - \frac{1}{\lambda+1} + \log\left(\frac{\lambda+1}{2\lambda+1}\right) \\ &> -\log 2. \end{aligned}$$

□

Lemma 8. *For $\lambda \geq 1$,*

$$\Psi'(\lambda+1) - 2\Psi'(2\lambda+1) < \frac{1}{\lambda+1} + \frac{1}{2(\lambda+1)^2} + \frac{1}{6(\lambda+1)^3} - \frac{2}{(2\lambda+1)} - \frac{1}{(2\lambda+1)^2} \tag{5}$$

and $\Psi(\lambda+1) - \Psi(2\lambda+1)$ is decreasing.

Proof. In [2], it is shown for $x > 0$,

$$\frac{1}{x} + \frac{1}{2x^2} < \Psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}$$

and inequality (5) follows immediately from this. The fact that $\Psi(\lambda+1) - \Psi(2\lambda+1)$ is decreasing follows from the fact that the right hand side of inequality (5) is

$$-\frac{6\lambda^3 + 20\lambda^2 + 23\lambda + 8}{6(\lambda+1)^3(2\lambda+1)^2}$$

and this quantity is clearly negative when λ is positive. □

The function $f_2(x,\lambda)$, $x=\cos\theta$, given in the following lemma, occurs in the proof of Theorem 5.

Lemma 9. *Define* $f_2 : (-1,1] \times [1,\infty) \to \mathbb{R}$ *by*

$$f_2(x,\lambda) = \frac{1}{\lambda} - \frac{2}{2\lambda+1} + \log 2 \\ + 2(\Psi(\lambda+1) - \Psi(2\lambda+1)) + \log(1+x) + (1-x)\frac{\lambda}{1+\lambda}. \tag{6}$$

For $\lambda \geq 1$, $f_2(1-1/\lambda, \lambda)$ *is decreasing.*

Proof. Let $g(\lambda) = f_2(1-1/\lambda,\lambda)$. Then using inequality (5) in Lemma 8 we have

$$\begin{aligned} g'(\lambda) &= -\frac{1}{\lambda^2} + \frac{4}{(2\lambda+1)^2} + 2(\Psi'(\lambda+1) - 2\Psi'(2\lambda+1)) + \frac{1}{\lambda(2\lambda-1)} - \frac{1}{(1+\lambda)^2} \\ &< -\frac{1}{\lambda^2} + \frac{2}{(2\lambda+1)^2} + \frac{2}{\lambda+1} + \frac{1}{3(\lambda+1)^3} - \frac{4}{2\lambda+1} + \frac{1}{\lambda(2\lambda-1)} \\ &= \frac{-24\lambda^6 - 46\lambda^5 - 23\lambda^4 + 22\lambda^3 + 35\lambda^2 + 18\lambda + 3}{3\lambda^2(2\lambda+1)^2(\lambda+1)^3(2\lambda-1)}. \end{aligned} \tag{7}$$

Clearly, the right hand side of (7) is negative for $\lambda \geq 1$. Thus, $f_2(1-1/\lambda,\lambda)$ is decreasing for $\lambda \geq 1$. □

Proof of Theorem 5. Let $\lambda \geq 1$ and $c_\lambda = 1 + 1/\lambda$. For this choice of c_λ, it is clear by Theorem 3 that $F(\cdot,\lambda)$ given by Eq. (3) is a convex univalent harmonic function in Δ for each $\lambda \geq 1$. We will use Theorem 6 to show F given by equation (3) is a weak subordination chain. Using differential equation (2), we can express F in terms of V_λ as

$$\begin{aligned} F(z,\lambda) &= \frac{\lambda V_\lambda(z) + (\lambda+1)zV'_\lambda(z)}{2\lambda+1} + \overline{\frac{\lambda V_\lambda(z) - (\lambda+1)zV'_\lambda(z)}{2\lambda+1}} \\ &= \frac{2\lambda}{2\lambda+1}\operatorname{Re}(V_\lambda(z)) + 2i\lambda\frac{\lambda+1}{2\lambda+1}\operatorname{Im}\left(\frac{z}{1+z} - \frac{1-z}{1+z}V_\lambda(z)\right). \end{aligned}$$

Clearly, $F(z,\cdot) \in \mathcal{C}^1[1,\infty)$ for each $z \in \Delta$ and $F(0,\lambda) = 0$ for each $\lambda \geq 1$. Since V_λ extends continuously into $\overline{\Delta}$ (see [7]), we can apply Theorem 6 to F.

To begin, we will first find a simplified expression for $[\partial F/\partial\lambda](e^{i\theta},\lambda)$, and the normal to $F(e^{i\theta},\lambda)$. From [7], it is known that

$$\operatorname{Re} V_\lambda\left(e^{i\theta}\right) = -\frac{1}{2} + 2^{\lambda-1}\frac{\Gamma^2(\lambda+1)}{\Gamma(2\lambda+1)}(1+\cos\theta)^\lambda, \quad \theta \in \mathbb{R}.$$

For $z = e^{i\theta}$,

$$\operatorname{Im}\left(\frac{z}{1+z} - \frac{1-z}{1+z}V_\lambda(z)\right) = \frac{\sin\theta}{1+\cos\theta}2^{\lambda-1}\frac{\Gamma^2(\lambda+1)}{\Gamma(2\lambda+1)}(1+\cos\theta)^\lambda.$$

Define

$$p(\theta,\lambda)=\frac{\lambda}{2\lambda+1}2^{\lambda}\frac{\Gamma^2(\lambda+1)}{\Gamma(2\lambda+1)}(1+\cos\theta)^{\lambda-1}.$$

Thus,

$$F\left(e^{i\theta},\lambda\right)=-\frac{\lambda}{2\lambda+1}+p(\theta,\lambda)\left(1+\cos\theta+i(\lambda+1)\sin\theta\right).$$

Therefore,

$$\begin{aligned}\frac{\partial F}{\partial\lambda}\left(e^{i\theta},\lambda\right)&=-\frac{1}{(2\lambda+1)^2}\\&\quad+p(\theta,\lambda)\Bigg[\bigg(\frac{1}{\lambda(2\lambda+1)}+\log 2+2(\Psi(\lambda+1)-\Psi(2\lambda+1))\\&\quad+\log(1+\cos\theta)\bigg)\bigg(1+\cos\theta+i(\lambda+1)\sin\theta\bigg)+i\sin\theta\Bigg].\end{aligned}\tag{8}$$

and

$$\frac{\partial}{\partial\theta}F\left(e^{i\theta},\lambda\right)=p(\theta,\lambda)\left(-i(\lambda^2-1)+i\lambda(\lambda+1)\cos\theta-\lambda\sin\theta\right).\tag{9}$$

To apply Theorem 6, it is equivalent to show

$$\frac{(1-\lambda^2+\lambda(1+\lambda)\cos\theta)^2+\lambda^2\sin^2\theta}{1+\lambda}\operatorname{Re}\left(\frac{i[\partial F/\partial\lambda](e^{i\theta},\lambda)}{[\partial/\partial\theta](F(e^{i\theta},\lambda))}\right)>0,\tag{10}$$

Letting $x=\cos\theta$ and for $x\in(-1,1]$, we can write the left side of (10) as $f_1(x,\lambda)+(1+x)f_2(x,\lambda)$ where $f_1:(-1,1]\times[1,\infty)\to\mathbb{R}$ is

$$f_1(x,\lambda)=\frac{\lambda-1-\lambda x}{\lambda(2\lambda+1)2^{\lambda}\dfrac{\Gamma^2(\lambda+1)}{\Gamma(2\lambda+1)}(1+x)^{\lambda-1}}$$

and f_2 is given by equation (6). Observe for $\lambda\geq 1$,

$$\lim_{x\to-1^+}\left(f_1(x,\lambda)+(1+x)f_2(x,\lambda)\right)=\infty.$$

Thus, to complete the proof of Theorem 5, we will show $f_1(x,\lambda)+(1+x)f_2(x,\lambda)>0$ for $-1<x\leq 1$ and $\lambda\geq 1$ via the following steps. See Figure 4 and Figure 5 for graphs of $f_1(x,\lambda)+(1+x)f_2(x,\lambda)$.

Step 1. Let $\lambda\geq 1$ and $-1<x\leq -1/2$. Set

$$G(x,\lambda)=(2\lambda+1)2^{\lambda}\frac{\Gamma^2(\lambda+1)}{\Gamma(2\lambda+1)}(1+x)^{\lambda-1}.$$

By Lemma 7

$$\begin{aligned}\frac{1}{G(x,\lambda)}\frac{\partial G}{\partial\lambda}(x,\lambda)&=\frac{2}{2\lambda+1}+\log 2+2(\Psi(\lambda+1)-\Psi(2\lambda+1))+\log(1+x)\\&\leq\frac{2}{3}+\log 2+2\log\left(\frac{2}{3}\right)+\log(1+x).\end{aligned}$$

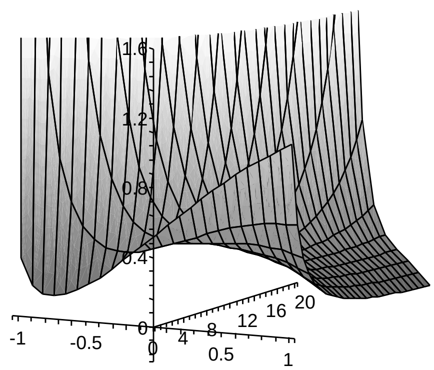

Figure 4: Above is the graph of $f_1(x, \lambda) + (1 + x)f_2(x, \lambda)$ for $x \in (-1, 1]$ and $\lambda \in [1, 20]$.

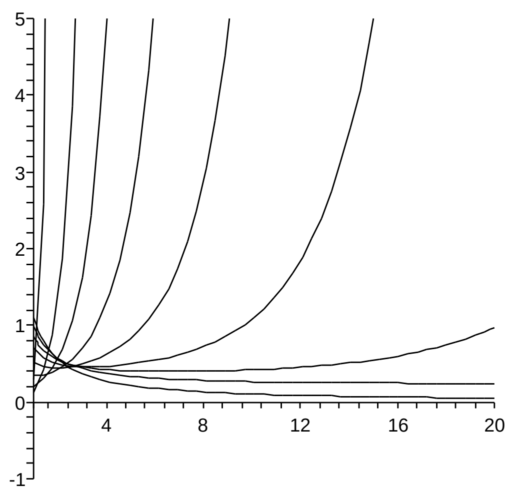

Figure 5: Above are the graphs of $f_1(x, \lambda) + (1 + x)f_2(x, \lambda)$ for $\lambda \in [1, 20]$ and fixed values of $x = -0.99, -3/4, -1/2, -1/4$, 0, 1/4, 1/2, 3/4, 1.

Therefore, if $\log(1+x) < -2/3 - \log(8/9)$ or equivalently if $x < (9/8)e^{-2/3} - 1 \approx -0.42$ fixed, $G(x,\lambda)$ is a decreasing function of λ. Thus,

$$f_1(x,\lambda) \geq \frac{\lambda - 1 - \lambda x}{\lambda G(x,1)} \geq -\frac{x}{3}.$$

Using $\lambda(1-x)/(1+\lambda) > \lambda(1+9/20)/(1+\lambda)$ for $\lambda \geq 1$ and $x \in (-1,-1/2]$ and Lemma 7,

$$f_1(x,\lambda) + (1+x)f_2(x,\lambda) > -\frac{x}{3} + (1+x)\left(\frac{1}{\lambda} - \frac{2}{1+\lambda} + \log 2 \right.$$
$$\left. +2\log\frac{\lambda+1}{2\lambda+1} + \log(1+x) + \frac{29}{20}\frac{\lambda}{1+\lambda}\right) \qquad (11)$$

Let $H(x,\lambda)$ be the expression on the righthand side above. We will prove $H(x,\lambda) \geq 0$ for $x \in (-1,-1/2]$ and $\lambda \geq 1$. First, observe

$$\frac{\partial H}{\partial \lambda}(x,\lambda) = \frac{1+x}{20}\left(\frac{58\lambda^3 - 71\lambda^2 - 80\lambda - 20}{\lambda^2(1+\lambda)^2(2\lambda+1)}\right).$$

Thus, for a fixed x, $H(x,\lambda)$ is a decreasing function of λ for $\lambda \in [1,2)$ and increasing for $\lambda \in (2,\infty)$. Next, let $x_0 = (25/18)e^{-22/15} - 1 \approx -0.68$. Then

$$\frac{\partial H}{\partial x}(x,2) = \frac{22}{15} + \log 2 + 2\log\frac{3}{5} + \log(1+x)$$

is negative for $-1 < x < x_0$ and positive for $x > x_0$. Hence $(x_0,2)$ gives an absolute minimum for H for $x \in (-1,-1/2]$ and $\lambda \geq 1$. To finish this case, it suffice to prove $H(x_0,2) \geq 0$. A simple calculation shows $H(x_0,2) = (1/3)(1-(25/6)e^{-22/15}) \geq 0$.

Step 2. Let $\lambda \geq 1$ and $-1/2 \leq x \leq 0$. Set

$$J(\lambda) = 2^\lambda \frac{\Gamma^2(\lambda+1)}{\Gamma(2\lambda+1)}.$$

By Lemma 7,

$$\frac{J'(\lambda)}{J(\lambda)} = \log 2 + 2(\Psi(\lambda+1) - \Psi(2\lambda+1)) \leq \log 2 + 2\log\left(\frac{2}{3}\right) < 0.$$

Therefore,

$$f_1(x,\lambda) = \frac{\lambda - 1 - \lambda x}{\lambda(2\lambda+1)J(\lambda)(1+x)^{\lambda-1}} \geq \frac{\lambda - 1 - \lambda x}{\lambda(2\lambda+1)}.$$

Thus, to complete this step, it suffices to show

$$K(x,\lambda) = \frac{\lambda - 1 - \lambda x}{\lambda(2\lambda+1)} + (1+x)q(x,\lambda) > 0$$

where

$$q(x,\lambda) = \frac{1}{\lambda} - \frac{2}{1+\lambda} + \log 2 + 2\log\left(\frac{\lambda+1}{2\lambda+1}\right) + \log(1+x) + (1-x)\frac{\lambda}{1+\lambda}. \quad (12)$$

We will show for a fixed $\lambda \geq 1$, $K(x,\lambda)$ is an increasing function of x when $-1/2 \leq x \leq 0$ and $K(-1/2,\lambda) > 0$ for $\lambda \geq 1$.

To do this, let $\lambda \geq 1$ be fixed and we see that

$$\frac{\partial K}{\partial x}(x,\lambda) = -\frac{1}{2\lambda+1} + q(x,\lambda) + (1+x)\frac{\partial q}{\partial x}(x,\lambda)$$

and

$$\frac{\partial^2 K}{\partial x^2}(x,\lambda) = 2\frac{\partial q}{\partial x}(x,\lambda) + (1+x)\frac{\partial^2 q}{\partial x^2}(x,\lambda) = \frac{1}{1+x} - \frac{2\lambda}{1+\lambda}.$$

Hence, $\partial^2 K/\partial x^2$ changes sign at most once for $-1/2 \leq x \leq 0$ and $\lambda \geq 1$. It is a simple calculation to show $[\partial^2 K/\partial x^2](-1/2,\lambda) > 0$ and $[\partial^2 K/\partial x^2](0,\lambda) \leq 0$ for $\lambda \geq 1$. Thus, $[\partial K/\partial x](x,\lambda)$ has at most one maximum in $-1/2 \leq x \leq 0$, and since $[\partial K/\partial x](-1/2,\lambda) > 0$ and $[\partial K/\partial x](0,\lambda) > 0$ for $\lambda \geq 1$, $K(x,\lambda)$ is an increasing function of x for $-1/2 \leq x \leq 0$ and $\lambda \geq 1$ fixed.

To complete the case with $-1/2 \leq x \leq 0$, by elementary calculus, we have $K(-1/2,\lambda)$ is a decreasing function for $\lambda \geq 1$ and $\lim_{\lambda\to\infty} K(-1/2,\lambda) > 0$. Thus, $K(-1/2,\lambda) > 0$ for $\lambda \geq 1$.

Step 3. Let $1 \leq \lambda \leq 2$ and $0 \leq x \leq 1 - 1/\lambda$. For $0 \leq x \leq 1 - 1/\lambda$, $f_1(x,\lambda) \geq 0$. Also, for $1 \leq \lambda \leq 2$, $1 - 1/\lambda \leq 1/\lambda$, and thus, $f_2(x,\lambda) \geq q(x,\lambda) \geq q(0,\lambda)$ where q is given by (12). Therefore, all that remains to be proved for this case is that $q(0,\lambda) > 0$. Define

$$P(\lambda) = \lambda^2(1+\lambda)^2(2\lambda+1)\frac{\partial q}{\partial \lambda}(0,\lambda) = 2\lambda^3 - 4\lambda^2 - 4\lambda - 1.$$

Then $P''(\lambda) > 0$ when $\lambda > 2/3$, and $P(1)$ and $P(2)$ are negative. Therefore, $P(\lambda) < 0$ for $1 \leq \lambda \leq 2$ and $q(0,\lambda)$ is a decreasing function of λ on this interval. Since $q(0,2) > 0.17$, we have the desired result.

Step 4. Let $1 \leq \lambda \leq 2$ and $1 - 1/\lambda \leq x \leq 1/\lambda$. Since $x \geq 1 - 1/\lambda$, $f_1(x,\lambda) \leq 0$. Also,

$$-\frac{1-\lambda+\lambda x}{(1+x)^{\lambda-1}} \geq -\frac{x}{(1+x)^{\lambda-1}} \geq -1,$$

Let

$$R(\lambda) = \lambda(2\lambda+1)2^\lambda \frac{\Gamma^2(\lambda+1)}{\Gamma(2\lambda+1)}.$$

Then

$$\frac{R'(\lambda)}{R(\lambda)} = \frac{4\lambda+1}{\lambda(2\lambda+1)} + \log 2 + 2(\Psi(\lambda+1) - \Psi(2\lambda+1))$$

which is decreasing for $1 \leq \lambda \leq 2$ by Lemma 8. Thus,

$$\frac{R'(\lambda)}{R(\lambda)} \geq \frac{R'(2)}{R(2)} > 0.42.$$

Therefore, since $R(\lambda) > 0$ for $1 \leq \lambda \leq 2$, R is increasing and we have

$$f_1(x,\lambda) = -\frac{1-\lambda+\lambda x}{(1+x)^{\lambda-1}}\frac{1}{R(\lambda)} \geq -\frac{1}{R(1)} = -\frac{1}{3}.$$

Since $f_2(x,\lambda)$ is a an increasing function of x for $x \leq 1/\lambda$, by Lemma 9,

$$\begin{aligned} f_1(x,\lambda) + (1+x)f_2(x,\lambda) &\geq -\frac{1}{3} + f_2(1-1/\lambda,\lambda) \\ &\geq -\frac{1}{3} + f_2(1/2,2) \\ &= -\frac{16}{15} + \log 3 \\ &> 0.03. \end{aligned}$$

Step 5. Let $\lambda \geq 2$ and $0 \leq x \leq 1/\lambda$. For these values of λ, $1/\lambda \leq 1 - 1/\lambda$ and $f_1(x,\lambda) \geq 0$. Also, $f_2(x,\lambda)$ is an increasing function of x when $x \leq 1/\lambda$. Therefore, to complete this step, it suffices to show $f_2(0,\lambda) > 0$. By Lemma 7, we have

$$\begin{aligned} f_2(0,\lambda) &= \frac{1}{\lambda} - \frac{2}{2\lambda+1} + \log 2 + 2(\Psi(\lambda+1) - \Psi(2\lambda+1)) + \frac{\lambda}{1+\lambda} \\ &\geq \frac{1}{\lambda} - \frac{2}{2\lambda+1} + \frac{\lambda}{\lambda+1} - \log 2. \end{aligned} \tag{13}$$

Define $S(\lambda)$ to be the right side of (13). Then $\lambda^2(2\lambda+1)^2(\lambda+1)^2 S'(\lambda) = 4\lambda^4 - 8\lambda^2 - 6\lambda - 1 > 0$ for $\lambda \geq 2$. Thus, $f_2(0,\lambda) \geq S(\lambda) \geq S(2) > 0.07$

Step 6. Let $\lambda \geq 2$ and $1/\lambda \leq x \leq 1 - 1/\lambda$. For these values of x, $f_1(x,\lambda) \geq 0$. Thus, it suffices to show $f_2(x,\lambda) > 0$. For these values of x, $f_2(x,\lambda)$ is a decreasing function of x. Hence $f_2(x,\lambda) \geq f_2(1-1/\lambda,\lambda)$ and by Lemma 9, $f_2(1-1/\lambda,\lambda)$ is decreasing. Since $\lim_{\lambda\to\infty} f_2(1-1/\lambda,\lambda) = 0$, we see $f_2(1-1/\lambda,\lambda) > 0$.

Step 7. Lastly, let $\lambda \geq 2$ and $1-1/\lambda \leq x \leq 1$. In this case, $f_1(x,\lambda)$ and $f_2(x,\lambda)$ are decreasing functions of x. Therefore, by Lemma 7, $f_2(x,\lambda) \geq f_2(1,\lambda) \geq 1/\lambda - 1/(2\lambda+1) > 0$. Thus,

$$\begin{aligned} f_1(x,\lambda) + (1+x)f_2(x,\lambda) &\geq f_1(1,\lambda) + \frac{1}{\lambda} - \frac{2}{2\lambda+1} \\ &= \frac{-2 + 4^\lambda \dfrac{\Gamma^2(\lambda+1)}{\Gamma(2\lambda+1)}}{\lambda(2\lambda+1)4^\lambda \dfrac{\Gamma^2(\lambda+1)}{\Gamma(2\lambda+1)}}. \end{aligned} \tag{14}$$

Define $T(\lambda)$ to be the numerator of the right side of (14). To complete this case, it suffices to show $T(\lambda) > 0$. By Lemma 7,

$$T'(\lambda) = 4^\lambda \frac{2\Gamma^2(\lambda+1)}{\Gamma(2\lambda+1)}(\log 2 + \Psi(\lambda+1) - \Psi(2\lambda+1)) > 0,$$

and so $T(\lambda) \geq T(2) = 2/3$.

Thus, for $\lambda \geq 1$ and $-1 < x \leq 1$, $f_1(x, \lambda) + (1 + x)f_2(x, \lambda) > 0$ and by Theorem 6, we have $F(\Delta, \lambda_1) \subset F(\Delta, \lambda_2)$ whenever $1 \leq \lambda_1 < \lambda_2$.

□

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