Harmonic Mappings Convex in One or Every Direction *

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Abstract

Given a convex complex-valued analytic mapping on the open unit disk in $\mathbb{C}$, we construct a family of complex-valued harmonic mappings convex in the direction of the imaginary axis. We also show that adding the condition of direction convexity preserving to the analytic mapping is a necessary and sufficient condition for the harmonic mapping to be convex. Using analytic radial slit mappings, we provide a three parameter family of harmonic mappings convex in the direction of the imaginary axis and show in some cases that as one parameter varies continuously, the mappings vary from being convex in the direction of the imaginary axis to being convex. In so doing, we also provide information on whether or not some analytic mappings are direction convexity preserving. Lastly, we will provide coefficient conditions leading to harmonic mappings which are starlike or convex of order α.

1 Introduction

A domain $D \subseteq \mathbb{C}$ is convex in the direction of φ, $\varphi \in [0, \pi)$, if every line parallel to the line through 0 and $e^{i\varphi}$ has a connected intersection with D, and a domain D that is convex in every direction is convex. In 1984, Clunie and Sheil-Small [2] introduced a technique for constructing a complex-valued harmonic mapping (one-to-one and sense-preserving, see Section 2) on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with a range that is convex in one or

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every direction. One interesting example that came from their work is the harmonic right half-plane mapping

$$\ell_0(z) = \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right) + \frac{1}{2}\overline{\left(\frac{z}{1-z} - \frac{z}{(1-z)^2}\right)}. \tag{1.1}$$

Writing $I(z) = z/(1-z)$, we can express ℓ_0 as

$$\ell_0(z) = \frac{I(z) + zI'(z)}{2} + \overline{\frac{I(z) - zI'(z)}{2}}.$$

In this paper, we generalize the structure seen above to generate harmonic mappings with a range that is convex in one or every direction without directly using the construction introduced in [2]. The generalization is done in the following way. For $f : \mathbb{D} \to \mathbb{C}$ a univalent analytic function with $f(0) = f'(0) - 1 = 0$, define

$$T_c[f](z) = \frac{f(z) + czf'(z)}{1+c} + \overline{\frac{f(z) - czf'(z)}{1+c}}, \qquad z \in \mathbb{D},\ c > 0. \tag{1.2}$$

Clearly, $T_1[I] = \ell_0$, and in [7], it was proved that $T_c[I]$ is a harmonic half-plane mapping of $\mathbb{D}$ onto $\{w \in \mathbb{C} : \operatorname{Re} w > -1/(1+c)\}$ for each $c > 0$. Another recent example of a harmonic mapping with the structure given in (1.2) uses Ruscheweyh and Suffridge's [11] continuous extension of the de la Vallée Poussin means $V_\lambda : \mathbb{D} \to \mathbb{C}$, $\lambda > 0$, defined by

$$V_\lambda(z) = \frac{\lambda z}{\lambda+1}\, {}_2F_1(1, 1-\lambda; 2+\lambda; -z) \tag{1.3}$$

where ${}_2F_1$ is the Gaussian hypergeometric function. In [7], $T_c[V_\lambda]$ was shown to be a harmonic mapping of $\mathbb{D}$ onto a convex domain for each $\lambda \geq 1/2$ and $c > 0$.

In the following sections, we investigate the mapping properties of $T_c[f]$ defined in (1.2). Our main theorem provides a necessary and sufficient condition on f such that $T_c[f]$ is a harmonic mapping of $\mathbb{D}$ onto a convex domain. We will also explore some coefficient conditions on f leading to other geometric properties of $T_c[f](\mathbb{D})$ and provide examples throughout to illustrate the interplay between the harmonic mapping $T_c[f]$ and the analytic mapping f.

2 Background

Let $\mathcal{H}_0(\mathbb{D})$ be the set of analytic functions on $\mathbb{D}$ that fix zero, and let $\mathcal{S} \subseteq \mathcal{H}_0(\mathbb{D})$ be the set of univalent functions with the added normalization $f'(0) = 1$. A harmonic function $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$ can be uniquely represented as $f = h + \overline{g}$ with $h, g \in \mathcal{H}_0(\mathbb{D})$. Furthermore, if we write $f(z) = f(x+iy) = u(x, y) + iv(x, y)$, then f is sense-preserving if the Jacobian, J_f, of the mapping $(x, y) \mapsto (u, v)$ is positive. The function f is locally univalent if J_f never vanishes on $\mathbb{D}$. By a result of Lewy [6], $f = h + \overline{g}$ is locally univalent and sense-preserving if and only if $|g'(z)| < |h'(z)|$ for all $z \in \mathbb{D}$. In this case, we simply say f is locally univalent. In addition, we call f univalent if f is one-to-one and sense-preserving on $\mathbb{D}$. Let $\mathcal{S}_H$ be the family of harmonic univalent functions on $\mathbb{D}$ of the form $f = h + \overline{g}$ normalized by $h(0) = g(0) = 0$ and $h'(0) = 1$. Clearly, $\mathcal{S} \subsetneq \mathcal{S}_H$.

To discuss the mapping properties of $T_c[f]$ defined in (1.2) we first need to introduce a number of families of analytic and harmonic functions. A domain D is close-to-convex if its complement can be written as the union of non-crossing half-lines. Let $\mathcal{C}$ and $\mathcal{C}_H$ denote the respective subclasses of $\mathcal{S}$ and $\mathcal{S}_H$ for which $f(\mathbb{D})$ is close-to-convex. Let $\mathcal{K}(\varphi)$ and $\mathcal{K}_H(\varphi)$ be the respective subclasses of $\mathcal{S}$ and $\mathcal{S}_H$ for which $f(\mathbb{D})$ is convex in the direction of φ. Note $\mathcal{K}(\varphi) \subseteq \mathcal{C}$ and $\mathcal{K}_H(\varphi) \subseteq \mathcal{C}_H$. If $\varphi = \pi/2$, we use the phrase convex in the direction of the imaginary axis. A domain D is starlike with respect to a point $w_0 \in D$ provided for every $w \in D$, the line segment $tw + (1-t)w_0, 0 \le t \le 1$ lies in D. Let $\mathcal{S}^*$ and $\mathcal{S}_H^*$ denote the respective subclasses of $\mathcal{S}$ and $\mathcal{S}_H$ for which $f(\mathbb{D})$ is starlike with respect to the origin. Lastly, let $\mathcal{K}$ and $\mathcal{K}_H$ denote the respective subclasses of $\mathcal{S}$ and $\mathcal{S}_H$ for which $f(\mathbb{D})$ is convex. We will simply say an analytic function f is close-to-convex, convex in the direction of φ, starlike, or convex if $f/f'(0)$ is in $\mathcal{C}$, $\mathcal{K}(\varphi)$, $\mathcal{S}^*$, or $\mathcal{K}$, respectively. Likewise, we will simply say a harmonic function $f = h + \overline{g}$ is close-to-convex, convex in the direction of φ, starlike, or convex if $f/h'(0)$ is in $\mathcal{C}_H$, $\mathcal{K}_H(\varphi)$, $\mathcal{S}_H^*$, or $\mathcal{K}_H$, respectively.

The following theorems due to Clunie and Sheil-Small [2] provide some information on the mapping properties of harmonic mappings and was used to construct the mapping ℓ_0 given in (1.1).

Theorem 2.1. *A harmonic $f = h + \overline{g}$ locally univalent on $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the imaginary axis if and only if $h + g$ is a conformal mapping of $\mathbb{D}$ onto a domain convex in the direction of the imaginary axis.*

Theorem 2.2. *A function $f = h + \overline{g} \in \mathcal{K}_H$ if and only if the analytic functions*

$$h(z) - e^{i\varphi} g(z), \quad \varphi \in [0, 2\pi)$$

are convex in the direction of $\varphi/2$ and f is suitably normalized. In this case, the functions

$$h(z) + \varepsilon g(z), \quad |\varepsilon| \le 1$$

are close-to-convex. In particular, h is close-to-convex.

The concept of direction convexity preserving will come into play for our main result. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic on $\mathbb{D}$. The function $f * g : \mathbb{D} \to \mathbb{C}$ defined as $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ is the Hadamard product of f and g. For $f \in \mathcal{S}$ and $F = H + \overline{G} \in \mathcal{S}_H$, define $f \tilde{*} F : \mathbb{D} \to \mathbb{C}$ as $f \tilde{*} F = f * H + \overline{f * G}$. A function $f \in \mathcal{S}$ is direction convexity preserving if it preserves the class $\mathcal{K}(\varphi)$, φ fixed, under the Hadamard product. Let DCP be the set of direction convexity preserving functions. It is well-known DCP $\subsetneq \mathcal{K}$. The following theorems are variations of results from [10] that provide useful characterizations of the family DCP.

Theorem 2.3. *Let $f \in \mathcal{H}_0(\mathbb{D})$. Then $f \in$ DCP if and only if for each $\gamma \in \mathbb{R}$, $f(z) + i\gamma z f'(z)$ is convex in the direction of the imaginary axis.*

Theorem 2.4. *Let $f \in \mathcal{K}$ be analytic on $\overline{\mathbb{D}}$ with $u(\theta) = \operatorname{Re} f(e^{i\theta})$ three times continuously differentiable. Then $f \in$ DCP if and only if*

$$u'(\theta)u'''(\theta) \le (u''(\theta))^2, \quad \theta \in \mathbb{R}. \tag{2.1}$$

The de la Vallée Poussin means are a well studied class of functions related to the family DCP. It is shown in [11] the continuous extension of the de la Vallée Poussin means given in (1.3) are in DCP for $\lambda \ge 1/2$. Additionally, in Remark 5.21 in [2], V_2 is used to show that not all functions in DCP preserve $\mathcal{K}_H(\varphi)$ under $\tilde{*}$. However, functions in DCP do preserve $\mathcal{K}_H$ under $\tilde{*}$ as stated in the following theorem from [10].

Theorem 2.5. *Let $g \in \mathcal{H}_0(\mathbb{D})$. Then $f \tilde{*} g \in \mathcal{K}_H$ for all $f \in \mathcal{K}_H$ if and only if $g \in$ DCP.*

Two other classes of functions that we will use in Section 5 are harmonic mappings which are starlike or convex of order α. A function $f \in \mathcal{S}_H$ is starlike of order α, $\alpha \in [0, 1)$, if

$$\frac{\partial}{\partial\theta} \arg f(re^{i\theta}) > \alpha, \quad 0 \le r < 1.$$

Denote this set of functions by $\mathcal{S}_H^*(\alpha)$. It is evident $\mathcal{S}_H^*(0) = \mathcal{S}_H^*$. A function $f \in \mathcal{S}_H$ is convex of order α, $\alpha \in [0, 1)$, if

$$\frac{\partial}{\partial\theta} \arg \left(\frac{\partial}{\partial\theta} f(re^{i\theta}) \right) > \alpha, \quad 0 \le r < 1. \tag{2.2}$$

Clearly the set of harmonic functions in $\mathcal{S}_H$ that are convex of order 0 is just $\mathcal{K}_H$. The following results of Jahangiri [4, 5] connect coefficient conditions to harmonic functions which are starlike or convex of order α.

Theorem 2.6. *Let $\alpha \in [0,1)$ and $f : \mathbb{D} \to \mathbb{C}$ be a harmonic function written as*

$$f(z) = \sum_{n=1}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}, \quad A_1 = 1. \tag{2.3}$$

If

$$\sum_{n=1}^{\infty} \left(\frac{n-\alpha}{1-\alpha} |A_n| + \frac{n+\alpha}{1-\alpha} |B_n| \right) \le 2, \tag{2.4}$$

then $f \in \mathcal{S}_H^*(\alpha)$. Moreover, if $A_n \le 0$ for $n \ge 2$ and $B_n \ge 0$ for $n \ge 1$, the condition given in (2.4) *is necessary for f to be in $\mathcal{S}_H^*(\alpha)$.*
If

$$\sum_{n=1}^{\infty} \left(\frac{n(n-\alpha)}{1-\alpha} |A_n| + \frac{n(n+\alpha)}{1-\alpha} |B_n| \right) \le 2, \tag{2.5}$$

then $f \in \mathcal{S}_H$ is convex of order α. Moreover, if $A_n \le 0$ for $n \ge 2$ and $B_n \le 0$ for $n \ge 1$, the condition given in (2.5) *is necessary for f to be convex of order α.*

3 Harmonic Mappings Convex in One or Every Direction

Our first lemma establishes precisely when $T_c[f]$ as defined by (1.2) is locally univalent.

Lemma 3.1. *The function $T_c[f]$ defined by* (1.2) *is locally univalent if and only if f is convex.*

Proof. Write $T_c[f] = F = H + \overline{G}$. Since F is locally univalent if and only if $|G'(z)| < |H'(z)|$ for all $z \in \mathbb{D}$, F will be locally univalent if and only if

$$\left| \frac{(1-c)f'(z) - czf''(z)}{(1+c)f'(z) + czf''(z)} \right| < 1$$

or equivalently, since $c > 0$ and $f \in \mathcal{S}$,

$$\left| \frac{1}{c} - \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \left| \frac{1}{c} + \left(1 + \frac{zf''(z)}{f'(z)} \right) \right|.$$

It can be easily seen that the inequality above is equivalent to

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0$$

which is the analytic condition for convexity. Thus, $T_c[f]$ is locally univalent if and only if f is convex. □

We immediately have the following.

Theorem 3.2. *The function $T_c[f]$ defined by* (1.2) *is convex in the direction of the imaginary axis if and only if f is convex.*

Proof. Again write $T_c[f] = H + \overline{G}$. Since $(H + G)(z) = 2f(z)/(1 + c)$, by Lemma 3.1 and Theorem 2.1, the proof is complete. □

The concept of direction convexity preserving provides the necessary and sufficient condition for $T_c[f]$ to be convex. We now state and prove our main theorem.

Theorem 3.3. *The function $T_c[f]$ defined by* (1.2) *is in $\mathcal{K}_H$ if and only if $f \in$ DCP.*

Proof. Let $I(z) = z/(1 - z)$. Then for $f \in \mathcal{S}$,

$$T_c[f](z) = (f \,\tilde{*}\, T_c[I])(z).$$

In [7], it was shown that $T_c[I]$ is a convex harmonic mapping of $\mathbb{D}$ onto the half-plane $\{w : \operatorname{Re} w > -1/(1 + c)\}$ for each $c > 0$. Hence, by Theorem 2.5, $T_c[f] \in \mathcal{K}_H$ if $f \in$ DCP.

To show $f \in$ DCP when $T_c[f] = H + \overline{G} \in \mathcal{K}_H$ we will use an approach similar to the proof of Theorem 2 in [10]. By Theorem 2.2 above, $T_c[f]$ is convex if and only if $ie^{-i\varphi/2}(H - e^{i\varphi}G)$ is convex in the direction of the imaginary axis for each $\varphi \in [0, 2\pi)$. Connecting this to f, we have for $\varphi \neq 0$ and $z \in \mathbb{D}$,

$$\begin{aligned} ie^{-i\varphi/2}(H(z) - e^{i\varphi}G(z)) &= -\frac{i\left(e^{i\varphi/2} - e^{-i\varphi/2}\right)}{1+c}\left(f(z) + c\frac{1+e^{i\varphi}}{1-e^{i\varphi}}zf'(z)\right) \\ &= \frac{2\sin(\varphi/2)}{1+c}\left(f(z) + c\frac{1+e^{i\varphi}}{1-e^{i\varphi}}zf'(z)\right). \end{aligned}$$

Write $c(1 + e^{i\varphi})/(1 - e^{i\varphi}) = i\gamma$ for $\gamma \in \mathbb{R}$. Since $\sin(\varphi/2)/(1 + c) \in \mathbb{R}^+$, from the equations above, $ie^{-i\varphi/2}(H - e^{i\varphi}G)$ is convex in the direction of the imaginary axis if and only if $f(z) + i\gamma zf'(z)$ is convex in the direction of the imaginary axis for each $\gamma \in \mathbb{R}$. Thus, by Theorem 2.3, $T_c[f] \in \mathcal{K}_H$ implies $f \in$ DCP. □

We have the following corollary as a consequence of Theorem 2.2.

Corollary 3.4. *For $|\varepsilon| \le 1$, $c > 0$, and $f \in$ DCP,*

$$(1+\varepsilon)f(z) + (1-\varepsilon)czf'(z)$$

is close-to-convex. In particular, $f(z) + czf'(z)$ is close-to-convex.

4 A Family of Harmonic Mappings Convex in One or Every Direction

In this section, we introduce a three parameter family of harmonic mappings which are convex in one or every direction constructed using $T_c[f]$ defined in (1.2). Using Theorem 3.3, we will also use the mapping properties of these harmonic functions to determine whether or not certain analytic mappings are direction convexity preserving.

Notice we may write $T_c[f]$ as

$$T_c[f](z) = \frac{2}{1+c}\operatorname{Re} f(z) + i\frac{2c}{1+c}\operatorname{Im}(zf'(z)). \tag{4.1}$$

and as we saw in the introduction, for $I(z) = z/(1-z)$, $T_1[I]$ is the harmonic half-plane mapping ℓ_0 given in (1.1). Observe $zI'(z) = z/(1-z)^2$ is the one-slit Koebe mapping. Generalizing this and utilizing the structure in equation (4.1), we consider the mapping properties of $T_c[f]$ for rotations of $f \in \mathcal{K}$ such that $zf'(z)$ is a radial n-slit map.

We begin with some background on the analytic functions we wish to use. For each $n \in \mathbb{N}$, let $g_n : \mathbb{D} \to \mathbb{C}$ be the radial n-slit mapping given by

$$g_n(z) = \frac{z}{(1-z^n)^{2/n}},$$

and for each $n \in \mathbb{N}$, let $f_n : \mathbb{D} \to \mathbb{C}$ be the solution of

$$zf_n'(z) = g_n(z), \quad f_n(0) = 0. \tag{4.2}$$

Each $f_n \in \mathcal{K}$ by Alexander's Theorem [1]. For the function g_n, the slits form an angle of $(2k+1)\pi/n$, $k = 0, \ldots, n-1$ with the positive real axis and the tips of the slits which are given by $g_n(e^{\pi i(2k+1)/n})$, $k = 0, \ldots, n-1$ lie on a circle of radius $2^{-2/n}$. From equation (4.2),

$$f_1(z) = \frac{z}{1-z}, \qquad f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z},$$

and

$$f_n(z) = z\,{}_2F_1\left(\frac{1}{n}, \frac{2}{n}; 1+\frac{1}{n}; z^n\right), \quad n \ge 3.$$

Clearly, f_1 maps $\mathbb{D}$ onto a right half-plane and f_2 maps $\mathbb{D}$ onto the strip $\{w \in \mathbb{C} : |\operatorname{Im} w| < \pi/4\}$. For $n \geq 3$, it is known f_n maps $\mathbb{D}$ onto the interior of a regular n-gon whose vertices form angles of $2\pi k/n$, $k = 0, \ldots, n-1$ with the positive real axis and lie on a circle with radius

$$\frac{2^{1-4/n}\Gamma^2(1/2 - 1/n)}{2n\sin(\pi/n)\Gamma(1-2/n)}$$

where Γ is the gamma function.

For $n \in \mathbb{N}$ and $\alpha \in [0, 2\pi/n)$, define $f_{n,\alpha} : \mathbb{D} \to \mathbb{C}$ by $f_{n,\alpha}(z) = e^{-i\alpha} f_n(e^{i\alpha} z)$. We will study the three parameter family of mappings given by

$$T_c[f_{n,\alpha}](z) = \frac{e^{-i\alpha} f_n(e^{i\alpha} z) + cz f_n'(e^{i\alpha} z)}{1+c} + \overline{\frac{e^{-i\alpha} f_n(e^{i\alpha} z) - cz f_n'(e^{i\alpha} z)}{1+c}}. \tag{4.3}$$

Note it is because of the rotational symmetry of f_n and g_n that it suffices to restrict α to the interval $[0, 2\pi/n)$ when determining the geometry of $T_c[f_{n,\alpha}](\mathbb{D})$. Immediately, we have the following corollary to Theorems 3.2 and 3.3.

Corollary 4.1. *For each $n \in \mathbb{N}$, $\alpha \in [0, 2\pi/n)$, and $c > 0$, the function $T_c[f_{n,\alpha}]$ given by (4.3) is in $\mathcal{K}_H(\pi/2)$. Moreover, $T_c[f_{n,\alpha}] \in \mathcal{K}_H$ if and only if $f_{n,\alpha} \in$ DCP.*

For $n = 1, 2$, we will explicitly determine $T_c[f_{n,\alpha}](\mathbb{D})$ below. However, for $n \geq 3$, this becomes more difficult because of the involvement of the hypergeometric function. Nonetheless, we will provide sufficient detail to conclude that $T_c[f_{n,\alpha}] \notin \mathcal{K}_H$ for $n \geq 3$ and thereby show the polygonal mappings $f_{n,\alpha}, n \geq 3$ are not in DCP by Corollary 4.1.

Before we begin the analysis for specific values of n, we make the following general observations. For $n \in \mathbb{N}$ and $\alpha \in [0, 2\pi/n)$, define $\varphi_{n,\alpha} : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_{n,\alpha}(k) = \frac{2\pi k}{n} - \alpha \tag{4.4}$$

For each $n \in \mathbb{N}$ and $\alpha \in [0, 2\pi/n)$, let $E_{n,\alpha} = \{e^{i\varphi_{n,\alpha}(k)} : k = 0, \ldots, n-1\}$. Then $T_c[f_{n,\alpha}]$ is continuous to $\partial\mathbb{D} \setminus E_{n,\alpha}$. Furthermore, $E_{n,\alpha}$ is the set of infinite boundary discontinuities of $T_c[f_{n,\alpha}]$. By equations (4.1) and (4.2), whenever $e^{-i\alpha} g_n(e^{i\alpha} z)$ has slit(s) which lie on the real axis, $T_c[f_{n,\alpha}]$ will collapse arc(s) of $\partial\mathbb{D}$ to single point(s) on the real axis. If n is odd, only one slit may lie on the real axis, and this occurs when $\alpha = 0$ or when $\alpha = \pi/n$. Specifically, if n is odd and $\theta \in (\varphi_{n,0}((n-1)/2), \varphi_{n,0}((n+1)/2))$,

$$T_c[f_{n,0}](e^{i\theta}) = -\frac{2}{1+c}\,{}_2F_1\left(\frac{1}{n}, \frac{2}{n}; 1+\frac{1}{n}; -1\right). \tag{4.5}$$

On the other hand, if n is odd and $\theta \in (\varphi_{n,\pi/n}(0), \varphi_{n,\pi/n}(1))$,

$$T_c[f_{n,\pi/n}](e^{i\theta}) = \frac{2}{1+c}\,{}_2F_1\left(\frac{1}{n}, \frac{2}{n}; 1+\frac{1}{n}; -1\right). \tag{4.6}$$

For n even, two slits may lie on the real axis and this occurs only for $\alpha = \pi/n$. Specifically, if n is even and $\theta \in (\varphi_{n,\pi/n}(0), \varphi_{n,\pi/n}(1))$,

$$T_c[f_{n,\pi/n}](e^{i\theta}) = \frac{2}{1+c}\,{}_2F_1\left(\frac{1}{n}, \frac{2}{n}; 1+\frac{1}{n}; -1\right). \tag{4.7}$$

and if $\theta \in (\varphi_{n,\pi/n}(n/2), \varphi_{n,\pi/n}(n/2+1))$,

$$T_c[f_{n,\pi/n}](e^{i\theta}) = -\frac{2}{1+c}\,{}_2F_1\left(\frac{1}{n}, \frac{2}{n}; 1+\frac{1}{n}; -1\right). \tag{4.8}$$

Additionally, for $n \geq 3$, f_n maps $\mathbb{D}$ onto the interior of a regular n-gon. Thus, by equation (4.1), for $n \geq 3$ and $z \in \mathbb{D}$,

$$\begin{aligned} \frac{2}{1+c}\min\{\operatorname{Re} f_{n,\alpha}(z) : z \in E_{n,\alpha}\} &< \operatorname{Re} T_c[f_{n,\alpha}](z) \\ &< \frac{2}{1+c}\max\{\operatorname{Re} f_{n,\alpha}(z) : z \in E_{n,\alpha}\}. \end{aligned} \tag{4.9}$$

To determine $T_c[f_{n,\alpha}](\mathbb{D})$ for $n = 1, 2$, we will perform a change of variables using

$$w = \frac{1+ze^{i\alpha}}{1-ze^{i\alpha}}. \tag{4.10}$$

Thus, as z varies in $\mathbb{D}$, we examine $T_c[f_{n,\alpha}](z(w)) = g(w) = u + iv$ as $w = x + iy$ varies in the right half-plane.

For $n = 1$,

$$T_c[f_{1,\alpha}](z) = \frac{2}{1+c}\operatorname{Re}\left(\frac{z}{1-e^{i\alpha}z}\right) + i\frac{2c}{1+c}\operatorname{Im}\left(\frac{z}{(1-e^{i\alpha}z)^2}\right)$$

and using (4.10) gives

$$u = \operatorname{Re} g(w) = \frac{1}{1+c}((x-1)\cos\alpha + y\sin\alpha) \tag{4.11}$$

and

$$v = \operatorname{Im} g(w) = \frac{c}{2(1+c)}(2xy\cos\alpha - (x^2 - y^2 - 1)\sin\alpha). \tag{4.12}$$

We will consider $\alpha = 0, \pi$ first as these values of α result in an arc of $\partial\mathbb{D}$ collapsing to a single point on the real axis. In this case, $T_c[f_{1,0}] = T_c[I]$ where again $I(z) = z/(1-z)$. In [7], it was shown that $T_c[I]$ maps $\mathbb{D}$ univalently onto $\{w \in \mathbb{C} : \operatorname{Re} w > -1/(1+c)\}$ for each $c > 0$ using the change of variables above. An argument similar to the one in [7] shows $T_c[f_{1,\pi}]$ maps $\mathbb{D}$ univalently onto the half-plane $\{w : \operatorname{Re} w < 1/(1+c)\}$. By directly showing $T_c[f_{1,0}], T_c[f_{1,\pi}] \in \mathcal{K}_H$, we have used a harmonic function approach to show the analytic functions $f_{1,0}, f_{1,\pi} \in$ DCP using Corollary 4.1. In this instance, since $f_{1,0}(z) = z/(1-z)$ is the identity under $*$ for functions in $\mathcal{H}_0(\mathbb{D})$, it is clearly in DCP and it is not difficult to show $f_{1,\pi}(z) = z/(1+z) \in$ DCP. However, it is interesting how a result regarding harmonic functions informs our understanding of analytic functions in this way.

For $\alpha \neq 0, \pi$ and $x = k \geq 0$ fixed, we see that $g = u + iv$ defined by equations (4.11) and (4.12) gives the parabola

$$v = \frac{c(1+c)}{2\sin\alpha}\left(u + \frac{\cos\alpha}{1+c}\right)^2 + \frac{c}{2(1+c)}(1 - k^2\csc^2\alpha)\sin\alpha \tag{4.13}$$

Thus, $T_c[f_{1,\alpha}](\partial\mathbb{D} \setminus \{e^{-i\alpha}\})$ is the parabola given by (4.13) when $k = 0$ and $T_c[f_{1,\alpha}]$ maps $\mathbb{D}$ to the exterior of this parabola. From this, we conclude $f_{1,\alpha} \notin$ DCP for $\alpha \neq 0, \pi$. See Figure 1 for an example.

For $n = 2$,

$$T_c[f_{2,\alpha}](z) = \frac{1}{1+c}\operatorname{Re}\left(e^{-i\alpha}\log\left(\frac{1+ze^{i\alpha}}{1-ze^{i\alpha}}\right)\right) + i\frac{2c}{1+c}\operatorname{Im}\left(\frac{z}{1-e^{2i\alpha}z^2}\right).$$

We will use the same change of variables in (4.10) to determine $T_c[f_{2,\alpha}](\mathbb{D})$ where

$$u = \operatorname{Re} g(w) = \begin{cases} \left(\cos\alpha\ln\sqrt{x^2+y^2} + \sin\alpha\tan^{-1}(y/x)\right)/(1+c) & \text{if } x > 0 \\ \left(\cos\alpha\ln y + (\pi/2)\sin\alpha\right)/(1+c) & \text{if } x = 0,\ y > 0 \\ \left(\cos\alpha\ln(-y) - (\pi/2)\sin\alpha\right)/(1+c) & \text{if } x = 0,\ y < 0 \end{cases} \tag{4.14}$$

and for $x, y \neq 0$

$$v = \operatorname{Im} g(w) = \frac{c}{2(1+c)}\left(\cos\alpha\left(y + \frac{y}{x^2+y^2}\right) - \sin\alpha\left(x - \frac{x}{x^2+y^2}\right)\right). \tag{4.15}$$

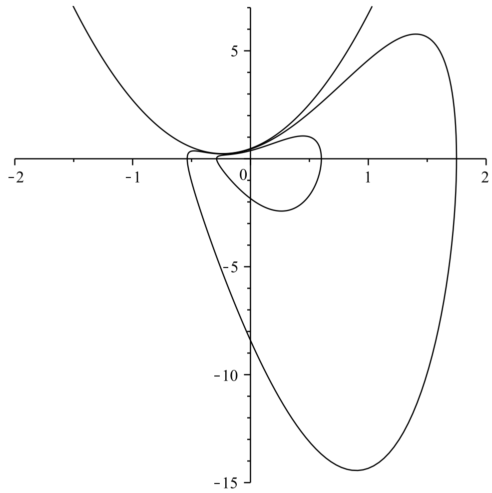

Figure 1: Graphs of $T_2[f_{1,\pi/4}](z)$ for $|z| = 0.5, 0.75$ and $|z| = 1, z \neq e^{-i\pi/4}$.

By (4.7) and (4.8), when $\alpha = \pi/2$, the two arcs of the unit circle $\{e^{i\theta} : \theta \in (-\pi/2, \pi/2)\}$ and $\{e^{i\theta} : \theta \in (\pi/2, 3\pi/2)\}$ collapse to the two points $(-\pi/(2(1+c)), 0)$ and $(\pi/(2(1+c)), 0)$, respectively. Clearly, by (4.14), for $\alpha = \pi/2$,

$$-\frac{\pi}{2(1+c)} < u < \frac{\pi}{2(1+c)}.$$

Also, for $x = k > 0$, fixed, $g = u + iv$ defined by equations (4.14) and (4.15) gives the curve

$$v = -\frac{c}{2k(1+c)}(k^2 - \cos^2((1+c)u)) \tag{4.16}$$

Thus, as k varies in $(0, \infty)$, we see that $T_c[f_{2,\pi/2}]$ is a strip mapping taking $\mathbb{D}$ onto $\{w : |\operatorname{Re} w| < \pi/(2(1+c))\}$, and by Corollary 4.1, we conclude the analytic mapping $f_{2,\pi/2}(z) = -i\log((1+iz)/(1-iz))$ is in DCP.

For $\alpha \neq \pi/2$, setting $x = 0$ in $g = u + iv$ defined by equations (4.14) and (4.15) shows $T_c[f_{2,\alpha}](\partial\mathbb{D} \setminus E_{2,\alpha})$ maps to the curves given by

$$v = \frac{c}{1+c}\cos\alpha\cosh\left(\frac{(1+c)u - \frac{\pi}{2}\sin\alpha}{\cos\alpha}\right), \quad y > 0 \tag{4.17}$$

$$v = -\frac{c}{1+c}\cos\alpha\cosh\left(\frac{(1+c)u + \frac{\pi}{2}\sin\alpha}{\cos\alpha}\right), \quad y < 0 \tag{4.18}$$

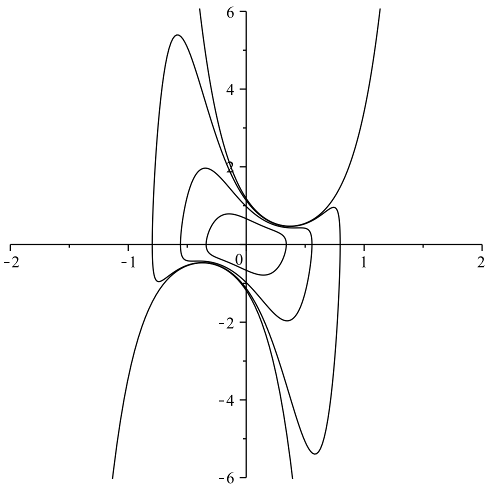

Figure 2: Graphs of $T_2[f_{2,\pi/4}](z)$, $|z| = 0.5, 0.75, 0.9$ and $T_2[f_{2,\pi/4}](\partial\mathbb{D} \setminus E_{2,\pi/4})$.

By Corollary 4.1, we know $T_c[f_{2,\alpha}](\mathbb{D})$ is convex in the direction of the imaginary axis. Furthermore, $T_c[f_{2,\alpha}]$ is continuous to $\partial\mathbb{D}\setminus E_{2,\alpha}$. Thus, $T_c[f_{2,\alpha}](\mathbb{D})$ fills the region between the curves given by equations (4.17) and (4.18), and $f_{2,\alpha} \notin$ DCP for $\alpha \neq \pi/2$. See Figure 2 for an example.

Recall for $n \geq 3$ and α fixed, $f_{n,\alpha}$ maps $\mathbb{D}$ onto the interior of a regular n-gon and thus, by equation (4.1), $\operatorname{Re} T_c[f_{n,\alpha}](\mathbb{D})$ is bounded as described in equation (4.9). However, $\operatorname{Im} T_c[f_{n,\alpha}](z)$ becomes unbounded as z approaches values in $E_{n,\alpha}$. Using the mapping properties of $f_{n,\alpha}$ and its derivative, equation (4.1) provides the following information about $T_c[f_{n,\alpha}](\partial\mathbb{D} \setminus E_{n,\alpha})$. First assume $\alpha \neq 0, \pi/n$ if n is odd and $\alpha \neq \pi/n$ if n is even, and let $k \in \{0, \ldots, n-1\}$ be fixed. If

$$\frac{n\alpha - \pi}{2\pi} < k < \frac{n\alpha + \pi(n-1)}{2\pi},$$

then $\operatorname{Re} T_c[f_{n,\alpha}](e^{i\theta})$ decreases for $\theta \in (\varphi_{n,\alpha}(k), \varphi_{n,\alpha}(k+1))$ where $\varphi_{n,\alpha}$ is given by (4.4) while $\operatorname{Im} T_c[f_{n,\alpha}](e^{i\theta})$ decreases for $\theta \in (\varphi_{n,\alpha}(k), \varphi_{n,\alpha}((2k+1)/2))$ and then increases for $\theta \in (\varphi_{n,\alpha}((2k+1)/2), \varphi_{n,\alpha}(k))$. For the remaining values of k, $\operatorname{Re} T_c[f_{n,\alpha}](e^{i\theta})$ increases for $\theta \in (\varphi_{n,\alpha}(k), \varphi_{n,\alpha}(k+1))$ while $\operatorname{Im} T_c[f_{n,\alpha}](e^{i\theta})$ increases for $\theta \in (\varphi_{n,\alpha}(k), \varphi_{n,\alpha}((2k+1)/2))$ and then decreases for $\theta \in (\varphi_{n,\alpha}((2k+1)/2), \varphi_{n,\alpha}(k+1))$. The relative extrema occur

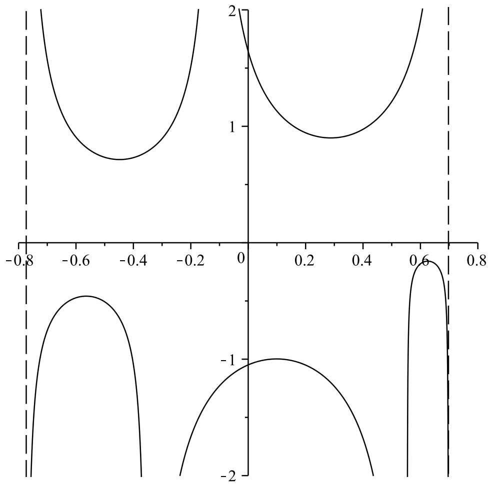

Figure 3: The solid curves above represent $T_2[f_{5,\pi/4}](\partial\mathbb{D} \setminus E_{5,\pi/4})$. These curves along with the dashed curves form $\partial T_2[f_{5,\pi/4}](\mathbb{D})$.

at the points

$$\left(\frac{2\cos\varphi_{n,\alpha}(k+1/2)}{1+c}\,{}_2F_1\left(\frac{1}{n},\frac{2}{n};1+\frac{1}{n};-1\right), \frac{2^{1-2/n}c}{1+c}\sin\varphi_{n,\alpha}(k+1/2)\right). \tag{4.19}$$

As in the $n = 2$ case, using the geometry given in Corollary 4.1, we see $T_c[f_{n,\alpha}]$ maps $\mathbb{D}$ onto the region trapped by the curves described above but now with the real part bounded by the values given in (4.9). See Figures 3 and 4 for examples.

For n odd and $\alpha = 0, \pi/n$ and for n even and $\alpha = \pi/n$, arc(s) of the unit circle now collapse to point(s) on the real axis. The above mapping behavior still holds with the exceptions given in (4.5), (4.6), (4.7), and (4.8). Compare Figures 3 and 4 with Figures 5 and 6 for examples with and without collapsing. Thus, we see for $n \geq 3$, the polygonal mappings $f_{n,\alpha} \notin$ DCP for any α.

As the rotation of a function in DCP is typically not again in DCP, looking at $T_c[f_{n,\alpha}]$ given by (4.3) with $c > 0$ fixed and $n = 1$ or $n = 2$, we obtain an interesting example of a family of harmonic mappings that as α varies continuously, the functions move between being convex in the direction of the imaginary axis and convex in every direction. It is worth noting that Pokhrel [8] introduced a subclass of DCP consisting of those

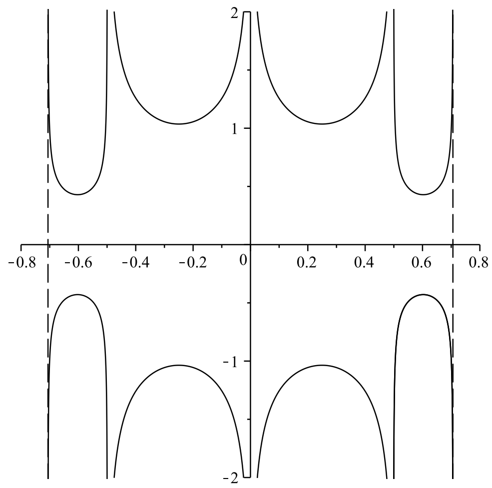

Figure 4: The solid curves above represent $T_2[f_{8,\pi/4}](\partial\mathbb{D} \setminus E_{8,\pi/4})$. These curves along with dashed curves form $\partial T_2[f_{8,\pi/4}](\mathbb{D})$.

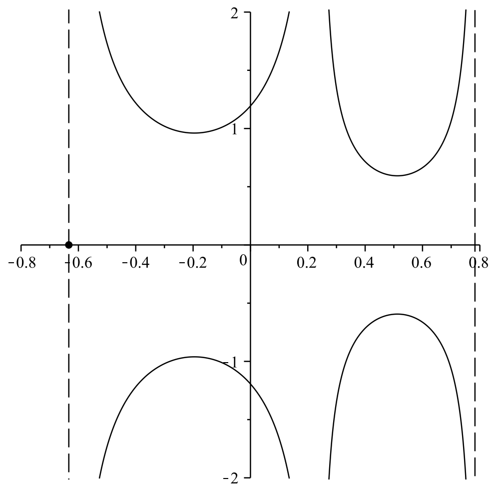

Figure 5: The solid curves and point on the real axis above represent $T_2[f_{5,0}](\partial\mathbb{D} \setminus E_{5,0})$. These along with dashed curves form $\partial T_2[f_{5,0}](\mathbb{D})$.

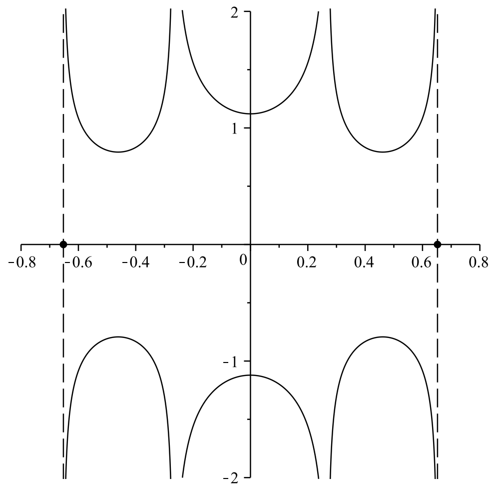

Figure 6: The solid curves and two points on the real axis above represent $T_2[f_{8,\pi/8}](\partial\mathbb{D} \setminus E_{8,\pi/8})$. These along with dashed curves form $\partial T_2[f_{8,\pi/8}](\mathbb{D})$.

functions in DCP which are rotation invariant. Using such functions, one could use a structure like that in (4.3) to construct other convex harmonic functions.

5 Coefficient Conditions for Harmonic Mappings Convex or Starlike of Order α

By Theorem 3.2, we know that if $f \in \mathcal{K}$, $T_c[f] \in \mathcal{K}_H(\pi/2) \not\subseteq \mathcal{S}_H^*$. In this section, we will use Theorem 2.4 to provide a coefficient condition on f that results in $T_c[f] \in \mathcal{S}_H^*(\alpha) \cap \mathcal{K}_H(\pi/2) \subsetneq \mathcal{C}_H$ or $T_c[f] \in \mathcal{S}_H$ being convex of order α. Thus, this latter condition also provides a criterion for f to be direction convexity preserving.

Theorem 5.1. *Let $\alpha \in [0, 1)$, $c \geq 1$, and $f : \mathbb{D} \to \mathbb{C}$ be an analytic with the form*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1. \tag{5.1}$$

If

$$\sum_{n=1}^{\infty} \frac{cn^2 - \alpha}{(1-\alpha)(1+c)} |a_n| \leq 1 \tag{5.2}$$

then $T_c[f] \in \mathcal{S}_H^*(\alpha)$ *and* $f \in \mathcal{K}$. *Furthermore, if* $a_n \leq 0$ *for* $n \geq 2$, *then* (5.2) *is necessary for* $T_c[f]$ *to be in* $\mathcal{S}_H^*(\alpha)$.

Proof. Using (5.1) and (1.2), we have

$$T_c[f](z) = \frac{1}{1+c}\sum_{n=1}^{\infty}(1+cn)a_n z^n + \frac{1}{1+c}\overline{\sum_{n=1}^{\infty}(1-cn)a_n z^n}.$$

Thus, substituting $A_n = (1+cn)a_n/(1+c)$ and $B_n = (1-cn)a_n/(1+c)$ into (2.4) gives (5.2) and by Theorem 2.6, $T_c[f] \in \mathcal{S}_H(\alpha)$. It follows then that $T_c[f]$ is locally univalent and thus, by Lemma 3.1, $f \in \mathcal{K}$.

Lastly, if $a_n \leq 0$ for $n \geq 2$, then (5.2) is necessary for $T_c[f] \in \mathcal{S}_H^*(\alpha)$ because then $A_n \leq 0$ for $n \geq 2$ and $B_n \geq 0$ for $n \geq 1$. □

As a corollary to this theorem, we use a starlike condition for harmonic functions to provide an alternate proof to Goodman's [3] classical result for analytic functions that states if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{H}_0(\mathbb{D})$ and $\sum_{n=2}^{\infty} n^2|a_n| \leq 1$, then $f \in \mathcal{K}$.

Corollary 5.2. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{H}_0(\mathbb{D})$ *and* $c \geq 1$. *If*

$$\sum_{n=2}^{\infty} n^2|a_n| \leq \frac{1}{c},$$

then $T_c[f] \in \mathcal{S}_H^*$ *and* $f \in \mathcal{K}$.

We also have the following.

Corollary 5.3. *Let* $c \geq 1$ *and suppose* $f \in \mathcal{S}$ *has the form* $f(z) = z - \sum_{n=2}^{\infty} c|a_n|z^n$. *Then* $f \in \mathcal{K}$ *if and only if* $T_c[f] \in \mathcal{S}_H^*$.

Proof. If $T_c[f] \in \mathcal{S}_H^*$, then $T_c[f]$ is locally univalent, and hence by Lemma 3.1, $f \in \mathcal{K}$. On the other hand, setting $B_n \equiv 0$ in inequality (2.5) in Theorem 2.6, we see that $f \in \mathcal{K}$ if and only if $\sum_{n=2}^{\infty} cn^2|a_n| \leq 1$. Thus, by Corollary 5.2, $f \in \mathcal{K}$ implies $T_c[f] \in \mathcal{S}_H^*$. □

Inequality (2.5) in Theorem 2.6 leads to the following theorem which provides a coefficient condition for $T_c[f]$ to be convex of order α and thus, by Theorem 3.3, also provides a condition for f to be in DCP. The proof of this theorem is similar to that of Theorem 5.1 and will be omitted.

Theorem 5.4. *Let $\alpha \in [0,1)$, $c \geq 1$, and $f \in \mathcal{K}$ be written as in* (5.1). *If*

$$\sum_{n=1}^{\infty} \frac{n(cn^2-\alpha)}{(1-\alpha)(1+c)}|a_n| \leq 1, \tag{5.3}$$

then $T_c[f]$ is convex of order α and $f \in$ DCP.

Interestingly, in the case of $\alpha = 0$ or $c \leq 1/\alpha - 1$, inequality (5.3) implies for $f_\varphi(z) = e^{-i\varphi} f(e^{i\varphi}z)$, $\varphi \in \mathbb{R}$, $T_c[f_\varphi]$ is a function in $\mathcal{K}_H$ for each φ (see [8]).

We will end this section with an application illustrating the results. Let $k = 2, 3, \ldots$ and $a \in \mathbb{C}$. Define $g_{k,a} : \mathbb{D} \to \mathbb{C}$ by

$$g_{k,a}(z) = z + az^k. \tag{5.4}$$

If $|a| \leq (1-\alpha)/(k^2-\alpha)$, then $g_{k,a}$ is convex of order α (see (2.5) with $B_n \equiv 0$) and by Theorem 5.1, $T_1[g_{k,a}] \in \mathcal{S}_H^*(\alpha)$. By Theorem 5.4, if $|a| \leq (1-\alpha)/(k(k^2-\alpha))$, then $T_1[g_{k,a}] \in \mathcal{S}_H$ is convex of order α and hence, $g_{k,a} \in$ DCP. Additionally, by a result in [8], $e^{i\varphi} g_{k,a} \in$ DCP for $\varphi \in \mathbb{R}$. So one could produce functions in $\mathcal{K}_H$ using rotations of $g_{k,a}$ as in done in (4.3).

In [8], it was shown $g_{3,a} \in$ DCP for $0 \leq a \leq 1/27$ and that this inequality is tight for any $a \geq 0$. Hence, $T_1[g_{3,a}] \in \mathcal{S}_H^* \setminus \mathcal{K}_H$ for $1/27 < a \leq 1/9$. See Figure 7. In fact, for $k \geq 3$, $T_1[g_{k,1/k^2}] \in \mathcal{S}_H^* \setminus \mathcal{K}_H$ and $T_1[g_{k,1/k^3}] \in \mathcal{K}_H$. However, for $k \geq 4$ and $a \geq 0$, $T_1[g_{k,a}]$ changes from a function in $\mathcal{S}_H^*$ to a function in $\mathcal{K}_H$ for some $a \in (1/k^3, 1/k^2)$. To see why $T_1[g_{k,1/k^2}] \notin \mathcal{K}_H$ if $k \geq 3$, we will use Theorem 2.4 to show $g_{k,1/k^2} \notin$ DCP.

Set $u(\theta) = \operatorname{Re} g_{k,1/k^2}(e^{i\theta}) = \cos\theta + (\cos k\theta)/k^2$. Then for $k \geq 3$, $3\pi/(2k) \in (0, \pi/2]$ and

$$\begin{aligned}\left(\left(u''\left(\frac{3\pi}{2k}\right)\right)^2 - u'\left(\frac{3\pi}{2k}\right)u'''\left(\frac{3\pi}{2k}\right)\right) &= 2 - \left(k + \frac{1}{k}\right)\sin\left(\frac{3\pi}{2k}\right) \\ &\leq 2 - k\sin\left(\frac{3\pi}{2k}\right)\end{aligned} \tag{5.5}$$

Let $p(k)$ be the last expression above. Then $p'(k) = 3\pi/(2k)\cos(3\pi/(2k)) - \sin(3\pi/(2k))$. Since $x\cos x - \sin x < 0$ for $x \in (0, \pi/2]$, we see that inequality (2.1) is violated when $\theta = 3\pi/(2k)$ and $g_{k,1/k^2} \notin$ DCP for $k \geq 3$.

6 Concluding Remarks

While proofs of the geometric properties of $T_c[f]$ defined in (1.2) rely on the results of Clunie and Sheil-Small [2], Theorem 3.3 provides an interesting

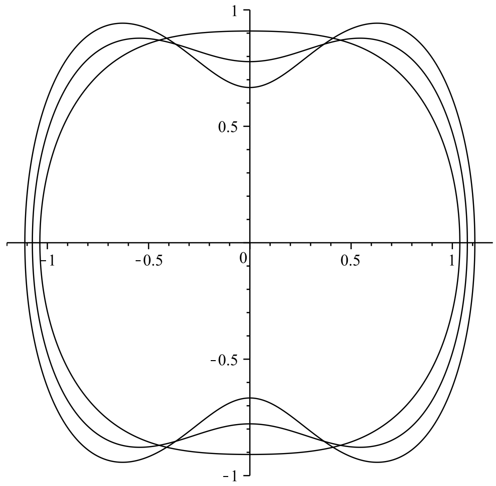

Figure 7: Above are the images of $\partial T_1[g_{3,1/9}](\mathbb{D})$, $\partial T_1[g_{3,2/27}](\mathbb{D})$, and $\partial T_1[g_{3,1/27}](\mathbb{D})$.

way of constructing convex harmonic mappings without directly using the shear construction. Moreover, both Lemma 3.1 and Theorem 3.3 present ways of using mapping properties of a harmonic function to determine if an analytic function is in $\mathcal{K}$ or DCP. In a recent paper, Ruscheweyh and Salinas [9] conjectured for $0 < a \leq b$ that ${}_1F_1(a; b; 1+z) \in$ DCP and perhaps this conjecture could be explored from a perspective of harmonic functions using Theorem 3.3. It would also be interesting to ask what subset of $\mathcal{K}$ leads to $T_c[f] \in \mathcal{S}_H^*$ and Theorem 3.3 and Theorem 5.1 give a partial answer to this. The subset of DCP that leads to $T_c[f]$ being convex of a certain order might be of interest too. It appears for $k \geq 4$, there exist α near one such that $T_1[g_{k,(1-\alpha)/(k^2-\alpha)}]$ is actually convex where $g_{k,a}$ is given by (5.4). Since $g_{k,(1-\alpha)/(k^2-\alpha)} \in \mathcal{S}^*(\beta)$ for some $\beta \geq \alpha$, perhaps the order of starlikeness of an analytic function f can be used to determine an order of convexity for the harmonic function $T_c[f]$.

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