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Convex Combinations of Planar Harmonic Mappings Realized Through Convolutions with Half-Strip Mappings

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Abstract Recent investigations into what geometric properties are preserved under the convolution of two planar harmonic mappings on the open unit disk $\mathbb{D}$ have typically involved half-plane and strip mappings. These results rely on having a convolution that is locally univalent and sense-preserving on $\mathbb{D}$, and thus, much focus has been on trying to satisfy this condition. We introduce a family of right half-strip harmonic mappings, $\Psi_c : \mathbb{D} \to \mathbb{C}$, $c > 0$, and consider the convolution $\Psi_c * f$ for a harmonic mapping $f = h + \overline{g} : \mathbb{D} \to \mathbb{C}$. We prove it is sufficient for $h \pm g$ to be starlike for $\Psi_c * f$ to be locally univalent and sense-preserving. Moreover, $\Psi_c * f$ decomposes into a convex combination of two harmonic mappings, one of which is f itself. This decomposition is key in addressing mapping properties of the convolution, and from it, we produce a family of convex octagonal harmonic mappings as well some other families of convex harmonic mappings. Additionally, motivated by the construction of Ψ_c, we introduce a generalized harmonic Bernardi integral operator. We demonstrate convolution preserving properties and a weak subordination relationship for this extended operator.

Keywords Harmonic mappings · Convolution · Convex in one direction · Convex · Starlike · Subordination

Mathematics Subject Classification 30C45

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1 Introduction

A domain $D \subseteq \mathbb{C}$ is convex in the direction of φ, $\varphi \in [0, \pi)$, if every line parallel to the line through 0 and $e^{i\varphi}$ has a connected intersection with D, and a domain D that is convex in every direction is convex. If for every w in a domain D, $tw \in D$ for $t \in [0, 1]$, the domain is starlike (with respect to the origin).

If $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ are analytic on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the function $(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n$ is called the Hadamard product or convolution of f and g. A harmonic function $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$ can be uniquely represented as $f = h + \overline{g}$ with h and g analytic on $\mathbb{D}$ and $h(0) = g(0) = 0$. Thus, in a similar manner, if $f = h + \overline{g}$ and $F = H + \overline{G}$ with $f(0) = F(0) = 0$ are harmonic on $\mathbb{D}$, we also call the function $f * F = h * H + \overline{g * G}$ the Hadamard product or convolution of f and F.

In 1984, Clunie and Sheil-Small [6] introduced what is now the well-known shear construction for producing a planar harmonic mapping (one-to-one and sense-preserving, see Sect. 2) on $\mathbb{D}$ with a range that is convex in one direction. One interesting example that came from their work is the harmonic right half-plane mapping $\ell_0 : \mathbb{D} \to \mathbb{C}$ defined as

$$\ell_0(z) = \frac{I(z) + zI'(z)}{2} + \overline{\frac{I(z) - zI'(z)}{2}} \tag{1.1}$$

where $I(z) = z/(1-z)$. While ℓ_0 is often considered the harmonic counterpart to the normalized analytic half-plane mapping I, we note there are infinitely many harmonic right half-plane mappings $f : \mathbb{D} \to \mathbb{C}$ with the normalization $f(0) = f_z(0) = 1$.

Further, as I acts as an identity under the convolution for functions analytic on $\mathbb{D}$ fixing zero, normalized right half-plane harmonic mappings defined on $\mathbb{D}$ became a focus of study for the convolution between two harmonic mappings. Even so, it is known that the harmonic convolution of one of these normalized right half-plane mappings with another harmonic mapping f with a convex range may not preserve properties of f (see [7]). However, for example, the convolution a normalized right half-plane mapping with another half-plane mapping or a strip mapping is known to be convex in the direction of the real axis provided the convolution is locally univalent and sense-preserving (see Sect. 2). Thus, recent research has involved finding conditions for which the convolution satisfies these latter conditions and these results often utilize the shear construction of Clunie and Sheil-Small. See [7, 8, 12, 14, 20–22] for some such results and for more details.

Consider the generalized half-plane mappings $I_c : \mathbb{D} \to \mathbb{C}$ defined as

$$I_c(z) = \frac{I(z) + czI'(z)}{1+c} + \overline{\frac{I(z) - czI'(z)}{1+c}}, \quad c > 0. \tag{1.2}$$

Notice $I_1 = \ell_0$. It was shown in [19] that I_c is one-to-one and sense-preserving for each $c > 0$ and $I_c(\mathbb{D}) = \{z \in \mathbb{C} : \operatorname{Re} z > -1/(1+c)\}$. For $c > 0$, we introduce the family of mappings $\Psi_c = H_c + \overline{G_c} : \mathbb{D} \to \mathbb{C}$ constructed from $I_c = h_c + \overline{g_c}$ so that $zH_c'(z) = h_c(z)$, $zG_c'(z) = -g_c(z)$, and $H_c(0) = G_c(0) = 0$. Thus,

$$\Psi_c(z) = \frac{cI(z) - \log(1-z)}{1+c} + \overline{\frac{cI(z) + \log(1-z)}{1+c}}, \quad c > 0. \tag{1.3}$$

We will show that Ψ_c is a half-strip mapping for each $c > 0$ and for a harmonic mapping $f = h + \overline{g} : \mathbb{D} \to \mathbb{C}$, $\Psi_c * f$ is locally univalent and sense-preserving whenever $(h \pm g)(\mathbb{D})$ is starlike, a distinct condition in comparison to other work regarding the local univalence of convolutions mentioned above. Moreover, $\Psi_c * f$ decomposes into a convex combination of two harmonic mappings, one of which is f itself. This decomposition is key in addressing mapping properties of the convolution, and from it, we produce a family of convex octagonal harmonic mappings as well some other families of convex harmonic mappings.

Let $\mathcal{K}$ and $\mathcal{S}^*$ denote the family of univalent analytic functions $f : \mathbb{D} \to C$ with a convex and starlike range, respectively, and satisfying $f(0) = f'(0) - 1 = 0$. The construction method for Ψ_c above is derived from a partial harmonic analogue to the classic 1915 theorem of Alexander [3]: $f \in \mathcal{S}^*$ if and only if $\int_0^z f(w)/w\, dw \in \mathcal{K}$. In 1990, Sheil-Small [25] extended this theorem to harmonic mappings as stated below.

Theorem A *If $f = h + \overline{g} : \mathbb{D} \to \mathbb{C}$ fixes zero, is univalent, and has a starlike range and if H and G are the analytic functions on $\mathbb{D}$ defined by*

$$zH'(z) = h(z),\ zG'(z) = -g(z),\ H(0) = G(0) = 0,$$

then $F = H + \overline{G}$ is univalent and has a convex range.

Other analytic generalizations of Alexander's theorem have also been studied, and in 1969, for a given analytic function $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = f'(0) - 1 = 0$, Bernardi [4] introduced the function $F_c : \mathbb{D} \to \mathbb{C}$ given by

$$F_c(z) = \frac{c+1}{z^c} \int_0^z \zeta^{c-1} f(\zeta)\, d\zeta, \quad c \in \mathbb{N}$$

and proved that F_c inherits some geometric properties of f. Motivated by the construction of Ψ_c, we extend this operator to planar harmonic mappings in the following way: For $f : \mathbb{D} \to \mathbb{C}$ a univalent harmonic mapping with $f(0) = f_z(0) - 1 = 0$, $\operatorname{Re} \lambda \geq 0$, and $\alpha \in \overline{\mathbb{D}}$, define the generalized harmonic Bernardi integral operator $\Lambda_{\lambda,\alpha}[f] : \mathbb{D} \to \mathbb{C}$ as

$$\Lambda_{\lambda,\alpha}[f](z) = \frac{\lambda+1}{z^\lambda} \int_0^z \zeta^{\lambda-1} h(\zeta)\, d\zeta + \alpha \overline{\frac{\lambda+1}{z^\lambda} \int_0^z \zeta^{\lambda-1} g(\zeta)\, d\zeta}. \tag{1.4}$$

Clearly, $\Lambda_{0,-1}$ is the harmonic mapping analogue to Alexander's operator given in Theorem A and so $\Lambda_{0,-1}[I_c] = \Psi_c$. Later in this paper, we provide some geometric results for this generalized harmonic Bernardi operator, including some convolution preserving properties that generalize results in [20] and some weak subordination (see Sect. 2) connections.

2 Additional Background

Let $\mathcal{H}_0(\mathbb{D})$ be the set of analytic functions on $\mathbb{D}$ that fix zero, and let $\mathcal{S} \subseteq \mathcal{H}_0(\mathbb{D})$ be the set of univalent functions with the added normalization $f'(0) = 1$. A harmonic function $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$ can be uniquely represented as $f = h + \overline{g}$ with $h, g \in \mathcal{H}_0(\mathbb{D})$. We call h the analytic part, g the coanalytic part, and $\omega = g'/h'$ the dilatation of f. Furthermore, if we write $f(z) = f(x+iy) = u(x, y)+iv(x, y)$, then f is sense-preserving if the Jacobian, J_f, of the mapping $(x, y) \mapsto (u, v)$ is positive. The function f is locally univalent if J_f never vanishes on $\mathbb{D}$, and $f = h+\overline{g}$ is locally univalent and sense-preserving if and only if $|g'(z)| < |h'(z)|$ for all $z \in \mathbb{D}$ [13]. In this case, we simply say f is locally univalent. In addition, we call f univalent if f is one-to-one and sense-preserving on $\mathbb{D}$. Let $\mathcal{S}_H$ be the family of harmonic univalent functions on $\mathbb{D}$ of the form $f = h+\overline{g}$ normalized by $h(0) = g(0) = 0$ and $h'(0) = 1$. Clearly, $\mathcal{S} \subsetneq \mathcal{S}_H$.

To discuss the mapping properties throughout the paper, it is useful to introduce a number of families of analytic and harmonic functions. A domain D is close-to-convex if its complement can be written as the union of non-crossing half-lines. Let $\mathcal{C}$ and $\mathcal{C}_H$ denote the respective subclasses of $\mathcal{S}$ and $\mathcal{S}_H$ for which $f(\mathbb{D})$ is close-to-convex. Let $\mathcal{K}(\varphi)$ and $\mathcal{K}_H(\varphi)$ be the respective subclasses of $\mathcal{S}$ and $\mathcal{S}_H$ for which $f(\mathbb{D})$ is convex in the direction of φ. Note $\mathcal{K}(\varphi) \subseteq \mathcal{C}$ and $\mathcal{K}_H(\varphi) \subseteq \mathcal{C}_H$. If $\varphi = \pi/2$ ($\varphi = 0$), we use the phrase convex in the direction of the imaginary (real) axis. Let $\mathcal{S}^*$ and $\mathcal{S}^*_H$ denote the respective subclasses of $\mathcal{S}$ and $\mathcal{S}_H$ for which $f(\mathbb{D})$ is starlike with respect to the origin. Lastly, let $\mathcal{K}$ and $\mathcal{K}_H$ denote the respective subclasses of $\mathcal{S}$ and $\mathcal{S}_H$ for which $f(\mathbb{D})$ is convex. We will simply say an analytic function f is close-to-convex, convex in the direction of φ, starlike, or convex if $f/f'(0)$ is in $\mathcal{C}$, $\mathcal{K}(\varphi)$, $\mathcal{S}^*$, or $\mathcal{K}$, respectively. Likewise, we will simply say a harmonic function $f = h + \overline{g}$ is close-to-convex, convex in the direction of φ, starlike, or convex if $f/h'(0)$ is in $\mathcal{C}_H$, $\mathcal{K}_H(\varphi)$, $\mathcal{S}^*_H$, or $\mathcal{K}_H$, respectively. If $f = h + \overline{g} \in \mathcal{S}_H$ has the added normalization that $g'(0) = 0$, then we write $f \in \mathcal{S}^0_H$, and we will attach similar superscript notation to all the subfamilies of $\mathcal{S}_H$ listed above for such normalized functions.

The Clunie and Sheil-Small shear construction for constructing harmonic mappings convex in one direction mentioned in the previous section is stated in the theorem below.

Theorem B *A function $f = h + \overline{g}$ locally univalent on $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of φ, $\varphi \in [0, \pi)$ if and only if the analytic function*

$$h(z) - e^{2i\varphi}g(z), \quad \varphi \in [0, \pi)$$

is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of φ.

In the case of two analytic functions $f_1, f_2 \in \mathcal{H}_0(\mathbb{D})$, f_1 is subordinate to f_2, written $f_1 \prec f_2$, if there exists a function $\omega \in \mathcal{H}_0(\mathbb{D})$, $|\omega(z)| < 1$ for $z \in \mathbb{D}$, such that $f_1(z) = f_2(\omega(z))$. If f_2 is univalent and $f_1(\mathbb{D}) \subseteq f_2(\mathbb{D})$, then f_1 is subordinate to f_2. However, for f, F harmonic on $\mathbb{D}$ with $f(0) = F(0)$, even if $f(\mathbb{D}) \subseteq F(\mathbb{D})$ and F is univalent, such an ω may not exist as is the case for analytic functions. Thusly, we say f is weakly subordinate to F if $f(\mathbb{D}) \subseteq F(\mathbb{D})$. See [19] for more details on weak subordination and [24] for general results on harmonic subordination.

3 Convex Combinations of Harmonic Mappings Realized through Convolutions with Half-Strip Mappings

Recall from the introduction that Ψ_c in Eq. (1.3) is constructed by applying Theorem A to the generalized half-plane mappings I_c in Eq. (1.2), or equivalently, $\Psi_c = \Lambda_{0,-1}[I_c]$ where $\Lambda_{0,-1}$ is defined in Eq. (1.4). We begin by establishing Ψ_c is indeed a half-strip mapping for each $c > 0$ and also provide a weak subordination result.

Theorem 3.1 *For each $c > 0$, Ψ_c as defined in Eq. (1.3) is a half-strip mapping in $\mathcal{S}_H$ and $(1/c)\Psi_c$ is weakly subordinate to the half-plane mapping I_c of Eq. (1.2).*

Proof The univalence of Ψ_c follows from the fact that $I_c \in \mathcal{K}_H$ and Theorem A. To determine $\Psi_c(\mathbb{D})$, we will perform a change of variables using $\zeta = (1+z)/(1-z)$, $z \in \mathbb{D}$. Thus, as z varies in $\mathbb{D}$, we examine $g(\zeta) = u + iv$ as $\zeta = x + iy$ varies in the right half-plane. Writing

$$\Psi_c(z) = \frac{2c}{1+c}\mathrm{Re}\left(\frac{z}{1-z}\right) + i\frac{2}{1+c}\mathrm{Im}(-\log(1-z))$$

and using the change of variables gives

$$u = \mathrm{Re}\, g(w) = \frac{c}{1+c}(x-1)$$

and

$$v = \mathrm{Im}\, g(w) = \frac{2}{1+c}\arctan\frac{y}{x+1}.$$

Observe the positive real axis $\{\zeta = x + iy : y = 0, x > 0\}$ is mapped monotonically onto the real axis from $-c/(1+c)$ to ∞. To find $\Psi_c(\mathbb{D}\backslash\{1\})$, set $x = 0$. Thus, $u = -c/(1+c)$, and v varies from $-\pi/(c+1)$ to $\pi/(c+1)$ as y varies from $-\infty$ to ∞. Next, let $x = k > 0$ be fixed. Then u is fixed at $c(k-1)/(c+1) > -c/(1+c)$ while again v varies from $-\pi/(c+1)$ to $\pi/(c+1)$ as y varies from $-\infty$ to ∞. Thus, $\Psi_c(\mathbb{D})$ is the half-strip $\{z \in \mathbb{C} : \mathrm{Re}\, z > -c/(1+c), |\mathrm{Im}\, z| < \pi/(c+1)\}$. Since $I_c(\mathbb{D}) = \{z \in \mathbb{C} : \mathrm{Re}\, z > -1/(1+c)\}$, the weak subordination result follows. □

The next convolution theorems are connected to the shear construction of Theorem B and are established in [7,8].

Theorem C *Let $f_1 = h_1 + \overline{g_1}$, $f_2 = h_2 + \overline{g_2} \in \mathcal{K}_H^0$. Suppose $(h_1 + g_1)(z) = z/(1-z)$ and either $(h_2 + g_2)(z) = z/(1-z)$ or*

$$(h_2 + g_2)(z) = \frac{1}{2i\sin\beta}\log\frac{1+ze^{i\beta}}{1+ze^{-i\beta}}, \quad \beta \in [\pi/2, \pi).$$

If $f_1 * f_2$ is locally univalent, then $f_1 * f_2 \in \mathcal{S}_H^0$ and is convex in the direction of the real axis.

Theorem D *Let $f_1 = h_1 + \overline{g_1} \in \mathcal{K}_H^0$ and suppose $(h_1 + g_1)(z) = z/(1-z)$. Let $\gamma \in [0, 2\pi)$ and suppose $f_2 = h_2 + \overline{g_2} \in \mathcal{S}_H^0$ maps $\mathbb{D}$ onto the slanted half-plane $H_\gamma = \{z \in \mathbb{C} : \mathrm{Re}(e^{i\gamma} z) > -1/2\}$. Then*

$$h_2(z) + e^{-2i\gamma} g_2(z) = \frac{z}{1 - ze^{i\gamma}}, \quad z \in \mathbb{D}.$$

Further, if $f_1 * f_2$ is locally univalent, then $f_1 * f_2 \in \mathcal{K}_H^0(-\gamma)$.

These theorems and others like them have motivated the recent work on harmonic convolutions centered around establishing conditions on f_1 and f_2 so that $f_1 * f_2$ is locally univalent. The theorems typically involve setting $f_1 = \ell_0$ given in Eq. (1.1) and then prescribing the dilatation of f_2. See [8,12,14] for examples of such results. Notice $\Psi_c = H_c + \overline{G_c}$ can also be thought of as a shear in the vertical direction by a positive real multiple of $I(z) = z/(1-z)$. That is, $(H_c + G_c)(z) = 2cI(z)/(1+c)$ and the proofs of the previous theorems can be adjusted to account for the positive constant $2c/(1+c)$. We now establish a local univalence result for the convolution $\Psi_c * f$ for $f \in \mathcal{S}_H$ that is of a different nature than those mentioned previously. This will then allow us to provide generalizations of Theorems C and D.

Theorem 3.2 *Let $f = h + \overline{g} \in \mathcal{S}_H$. If both $h + g$ and $h - g$ are starlike, then $\Psi_c * f$ is locally univalent.*

In order to prove this theorem, we need the following from [15].

Theorem E *Let $N, D : \mathbb{D} \to \mathbb{C}$ be analytic with $N(0) = D(0) = 0$. If D maps $\mathbb{D}$ onto a (possibly many-sheeted) domain which is starlike with respect to the origin, then* $\mathrm{Re}(N'(z)/D'(z)) > 0$ *implies* $\mathrm{Re}(N(z)/D(z)) > 0$.

Proof of Theorem 3.2 Let $f = h + \overline{g} \in \mathcal{S}_H$. Then

$$(\Psi_c * f)(z) = \frac{ch(z) + \int_0^z \frac{h(t)}{t}\, dt}{1+c} + \overline{\frac{cg(z) - \int_0^z \frac{g(t)}{t}\, dt}{1+c}}. \tag{3.1}$$

Since h is analytic on $\mathbb{D}$, $h'(0) = 1$, and $c > 0$, $czh'(z) + h(z)$ is not identically zero on $\mathbb{D}$. Further, since g is analytic on $\mathbb{D}$, $\omega(z) = (czg'(z) - g(z))/(czh'(z) + h(z))$ is analytic where defined on $\mathbb{D}$ and any singularity will be isolated. For all $z \in \mathbb{D}$ for which ω is defined, we will show $|\omega(z)| < 1$ which in turn will show ω has only removable singularities on $\mathbb{D}$. Finally, we may define ω on all of $\mathbb{D}$ and by the Maximum Principle, conclude $|\omega(z)| < 1$ for $z \in \mathbb{D}$ which proves $\Psi_c * f$ is locally univalent.

For all $z \in \mathbb{D}$ for which ω is defined, $|\omega(z)| < 1$ if and only if

$$\mathrm{Re}\, \frac{cz(h+g)'(z) + (h-g)(z)}{cz(h-g)'(z) + (h+g)(z)} > 0. \tag{3.2}$$

Since $h \pm g$ are starlike and $c > 0$, $\operatorname{Re}(cz(h \pm g)'(z)/(h \pm g)(z)) > 0$. Further, since f is locally univalent $|g'(z)/h'(z)| < 1$ and hence $\operatorname{Re}((h-g)'(z)/(h+g)'(z)) > 0$. Since $h+g$ is starlike, by Theorem E, this last inequality implies $\operatorname{Re}(h-g)(z)/(h+g)(z) > 0$. Collectively, we have

$$\operatorname{Re}\frac{(h+g)(z)+cz(h-g)'(z)}{cz(h+g)'(z)} > 0$$

and

$$\operatorname{Re}\frac{(h+g)(z)+cz(h-g)'(z)}{(h-g)(z)} > 0$$

which together show inequality (3.2) is satisfied. □

Similar calculations immediately give the next corollary.

Corollary 3.3 *Let $f = h + \overline{g} \in \mathcal{S}_H$ and I_c be the half-plane mapping given in Eq. (1.2). If both $h+g$ and $h-g$ are starlike, then $\Lambda_{0,\alpha}[I_c] * f$ is locally univalent for each $|\alpha| \le 1$ where $\Lambda_{0,\alpha}$ is defined in Eq. (1.4).*

As a consequence of the local univalence criteria of Theorem 3.2, we have the following generalizations of Theorems C and D.

Theorem 3.4 *Let $f = h + \overline{g} \in \mathcal{K}_H^0$. If $(h+g)(z) = z/(1-z)$ or $(h+g)(z) = 1/(2i \sin\beta) \log((1+ze^{i\beta})/(1+ze^{-i\beta}))$, $\beta \in [\pi/2, \pi)$ and $h-g$ is starlike, then $\Psi_c * f \in \mathcal{S}_H^0$ and is convex in the direction of the real axis.*

Theorem 3.5 *Let $f = h+\overline{g} \in \mathcal{K}_H^0$. If $h(z)+e^{-2i\gamma}g(z) = z/(1-e^{i\gamma}z)$, $\gamma \in [0, 2\pi)$ and $h-g$ is starlike, then $\Psi_c * f \in \mathcal{K}_H^0(-\gamma)$.*

While we used Theorem A to conclude $\Psi_c \in \mathcal{K}_H$, we remark the converse to Theorem A fails [25]. However, in 2014 Ponnusammy and Kaliraj [22] proved the following version of a converse with an additional restriction.

Theorem F *Suppose $F = H + \overline{G} \in \mathcal{K}_H$ and $DF(z) = zH'(z) - \overline{zG'(z)}$ is locally univalent on $\mathbb{D}$. Then the harmonic function DF is univalent and starlike on $\mathbb{D}$.*

Because of the connection between Ψ_c and the half-plane mapping I_c, we have the following.

Theorem 3.6 *Let I_c be as given in Eq. (1.2) and $f \in \mathcal{S}_H$. For each $c > 0$, if $I_c * f$ is starlike, then $\Psi_c * f \in \mathcal{K}_H$. For each $c > 0$, if $I_c * f$ is locally univalent and $\Psi_c * f$ is convex, then $I_c * f \in \mathcal{S}_H^*$.*

Proof Observe

$$\Psi_c(z) = -\log(1-z) * \frac{I(z)+czI'(z)}{1+c} + \overline{\log(1-z) * \frac{I(z)-czI'(z)}{1+c}}. \quad (3.3)$$

Hence, for $f = h + \overline{g} \in S_H$,

$$(\Psi_c * f)(z) = -\log(1-z) * \frac{h(z) + czh'(z)}{1+c} + \overline{\log(1-z) * \frac{g(z) - czg'(z)}{1+c}}$$
$$= \Lambda_{0,-1}[I_c * f](z).$$

Thus, the theorem follows from Theorems A and F. □

Many of the convolution results for harmonic mappings have only been able to conclude the convolution is convex in a given direction, and indeed by noting that the dilatation of the half-plane mapping I_c is

$$\omega_c(z) = -\frac{z + \frac{c-1}{c+1}}{1 + \frac{c-1}{c+1}z}$$

we may use a modification of Theorem 4 in [8] to conclude $\ell_0 * I_c$ is convex in the direction of the real axis if $c > 1$. Because of Theorem 3.6, we can now broaden this result to show $\ell_0 * I_c$ is actually a starlike mapping for all $c \geq 1$.

Corollary 3.7 *Let I_c and Ψ_c be as given by Eqs.* (1.2) *and* (1.3). *The mappings $\Psi_c * I_d$, $c, d > 0$ are half-plane mappings and $I_1 * I_c = \ell_0 * I_c$, $c \geq 1$ is a starlike mapping.*

Proof Write $I_d = h_d + \overline{g_d}$. Since $h_d \pm g_d$ are both starlike, Theorem 3.2 shows $\Psi_c * I_d$ is locally univalent. Using the same change of variables as in Theorem 3.1, one can show that $\Psi_c * I_d$ maps $\partial\mathbb{D}\backslash\{1\}$ to the line segment with real part equal to $-(d+c)/((1+d)(1+c))$ and the imaginary part ranging from $-\pi/((1+c)(1+d))$ to $\pi/((1+c)(1+d))$ and $\mathbb{D}$ is mapped onto the right half-plane $\{z : \operatorname{Re} z > -(d+c)/((1+d)(1+c))\}$. See Fig. 1. By the comments preceding this corollary, $\ell_0 * I_c$ is univalent and convex in the direction of the real axis provided $c > 1$ and $\ell_0 * \ell_0$ is univalent and convex in the direction of the real axis by Theorem 3 in [8]. Thus, since $\ell_0 * I_c$ is locally univalent for $c \geq 1$, by Theorem 3.6, $I_1 * I_c = \ell_0 * I_c$ is indeed starlike for each $c \geq 1$. □

Next, we show how the convex combination of harmonic mappings can be realized through the convolution of the half-strip mapping Ψ_c with $f \in \mathcal{S}_H$.

Theorem 3.8 *Let $f \in \mathcal{S}_H$ and Ψ_c be the half-strip mapping given in Eq.* (1.3). *Then*

$$(\Psi_c * f)(z) = \frac{c}{1+c} f(z) + \frac{1}{1+c}\Lambda_{0,-1}[f](z). \tag{3.4}$$

where $\Lambda_{0,-1}$ is defined in Eq. (1.4).

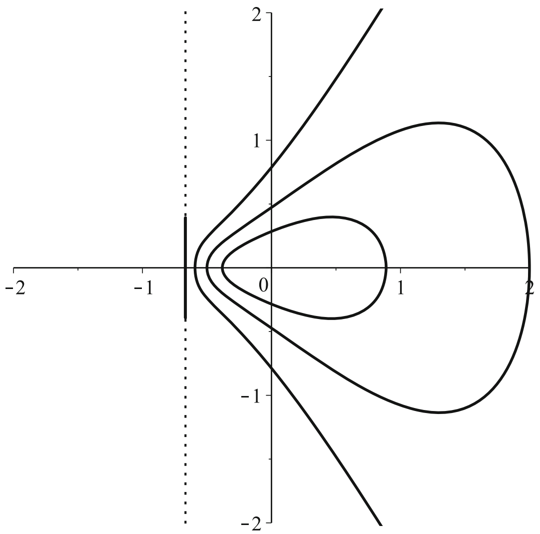


Fig. 1 Images of $(\Psi_{1/3} * I_5)(|z| = r)$ for $r = 0.4, 0.6, 0.8$, $(\Psi_{1/3} * I_5)(\partial\mathbb{D}\backslash\{1\})$, and the cluster set of $\Psi_{1/3} * I_5$ at 1 (*dotted*)

Proof We observe

$$\begin{aligned}(\Psi_c * f)(z) &= \frac{ch(z) + \int_0^z \frac{h(t)}{t}\,dt}{1+c} + \overline{\frac{cg(z) - \int_0^z \frac{g(t)}{t}\,dt}{1+c}} \\ &= \frac{c}{1+c}\left(h(z) + \overline{g(z)}\right) + \frac{1}{1+c}\left(\int_0^z \frac{h(t)}{t}\,dt - \overline{\int_0^z \frac{g(t)}{t}\,dt}\right) \\ &= \frac{c}{1+c} f(z) + \frac{1}{1+c}\Lambda_{0,-1}[f](z).\end{aligned}$$

Thusly, the convolution $\Psi_c * f$ decomposes into a convex combination of $\Lambda_{0,-1}[f]$ and f. □

By Theorem A, if $f \in \mathcal{K}_H$, $\Psi_c * f$ is realized as a convex combination of two functions in $\mathcal{K}_H$. On the other hand, if $f \in \mathcal{S}_H^*$, $\Psi_c * f$ is a convex combination of a function in $\mathcal{S}_H^*$ and in $\mathcal{K}_H$. In particular, by Corollary 3.7, we remark $\Psi_c * I_d$ is a convex combination of a half-strip mapping and a half-plane mapping that resulted in another half-plane mapping. This convex combination structure is central to developing several families of convex harmonic maps, and we begin with a family of convex octagonal mappings.

Corollary 3.9 *Let* $f_1 = h_1 + \overline{g_1} \in \mathcal{S}_H^0$ *be the mapping such that*

$$h_1(z) + g_1(z) = -\frac{i}{2}\log\frac{1+iz}{1-iz}$$

and $g_1'(z)/h_1'(z) = -z^2$. *Then for each* $c > 0$, $\Psi_c * f_1$ *is a convex mapping of* $\mathbb{D}$ *onto an octagon.*

Proof A calculation shows

$$f_1(z) = \frac{1}{4}\log\frac{1+z}{1-z} - \frac{i}{4}\log\frac{1+iz}{1-iz} - \left(\overline{\frac{1}{4}\log\frac{1+z}{1-z} + \frac{i}{4}\log\frac{1+iz}{1-iz}}\right). \tag{3.5}$$

This function sends $\mathbb{D}$ onto a square with vertices at $\pm\pi/4 + i\pi/4$ and $\pm\pi/4 - i\pi/4$. See Fig. 2 and [9] for more details on the mapping properties of f_1. By Eq. (3.5), $(h_1 - g_1)(z) = 1/2\log((1+z)/(1-z))$ which is a convex mapping. Hence, by Theorem 3.4, $\Psi_c * f_1$ is univalent and convex in the direction of the real axis for each $c > 0$.

In order to use the convex combination decomposition of Theorem 3.8, we need to establish the behavior of $F_1 = \Lambda_{0,-1}[f_1]$, which we know to be convex by Theorem A. We will show F_1 also maps onto a square and its vertices are at $\pm\pi^2/8$ and $\pm i\pi^2/8$. Hence, $\Lambda_{0,-1}[f_1](\mathbb{D})$ is a rotation of $f_1(\mathbb{D})$ by $\pi/2$. A calculation gives

$$F_1(z) = \frac{1}{2}\text{Re}\,(\text{Li}_2(z) - \text{Li}_2(-z)) + i\frac{1}{2}\text{Im}\,(i(\text{Li}_2(-iz) - \text{Li}_2(iz))) \tag{3.6}$$

where Li_2 denotes the dilogarithm function given by the series $\text{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2$, $|z| \le 1$. Through a Fourier series calculation, if $\theta \in [0, 2\pi)$,

$$\begin{aligned} F_1(e^{i\theta}) &= \sum_{n=0}^{\infty}\frac{\cos((2n+1)\theta)}{(2n+1)^2} + i\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}\sin((2n+1)\theta)}{(2n+1)^2} \\ &= \begin{cases} \frac{\pi}{4}\left(-\theta + \frac{\pi}{2} - i\theta\right) & \text{if } \theta \in [0, \pi/2], \\ \frac{\pi}{4}\left(-\theta + \frac{\pi}{2} + i\,(-\theta + \pi)\right) & \text{if } \theta \in [\pi/2, \pi], \\ \frac{\pi}{4}\left(\theta - \frac{3\pi}{2} + i\,(-\theta + \pi)\right) & \text{if } \theta \in [\pi, 3\pi/2], \\ \frac{\pi}{4}\left(\theta - \frac{3\pi}{2} + i\,(\theta - 2\pi)\right) & \text{if } \theta \in [3\pi/2, 2\pi). \end{cases} \end{aligned} \tag{3.7}$$

Each of the four pieces above describes a side of the square with vertices at $\pm\pi^2/8$ and $\pm i\pi^2/8$. See Fig. 3.

By Theorem 3.8, the formulas in Eq. (3.7), and the mapping properties of f_1, for $\theta \in (0, \pi) \cup (\pi, 2\pi)$

$$\begin{aligned} &(\Psi_c * f_1)(e^{i\theta}) \\ &= \begin{cases} \frac{\pi}{4}\frac{1}{1+c}\left(\left(c - \theta + \frac{\pi}{2}\right) + i\,(c + \theta)\right) & \text{if } \theta \in [0, \pi/2], \\ \frac{\pi}{4}\frac{1}{1+c}\left(\left(-c - \theta + \frac{\pi}{2}\right) + i(c - \theta + \pi)\right) & \text{if } \theta \in [\pi/2, \pi), \\ \frac{\pi}{4}\frac{1}{1+c}\left(\left(-c + \theta - \frac{3\pi}{2}\right) + i(-c - \theta + \pi)\right) & \text{if } \theta \in (\pi, 3\pi/2], \\ \frac{\pi}{4}\frac{1}{1+c}\left(\left(c + \theta - \frac{3\pi}{2}\right) + i(-c + \theta - 2\pi)\right) & \text{if } \theta \in [3\pi/2, 2\pi). \end{cases} \end{aligned} \tag{3.8}$$

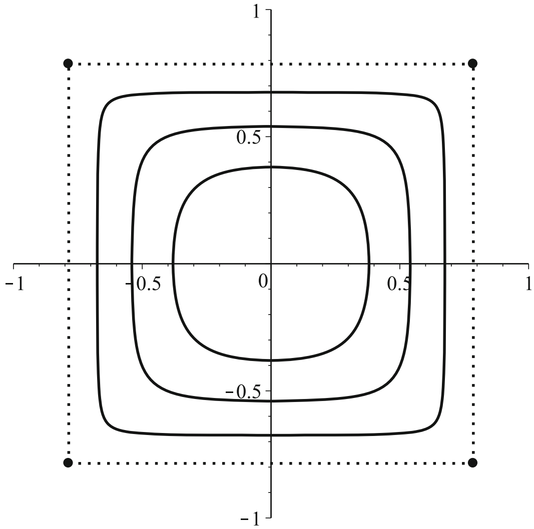


Fig. 2 Images of $f_1(|z| = r)$ for $r = 0.4, 0.6, 0.8$, $f_1(\partial\mathbb{D}\backslash\{\pm 1, \pm i\})$ and the cluster set of f_1 at $\pm 1, \pm i$ (*dotted*)

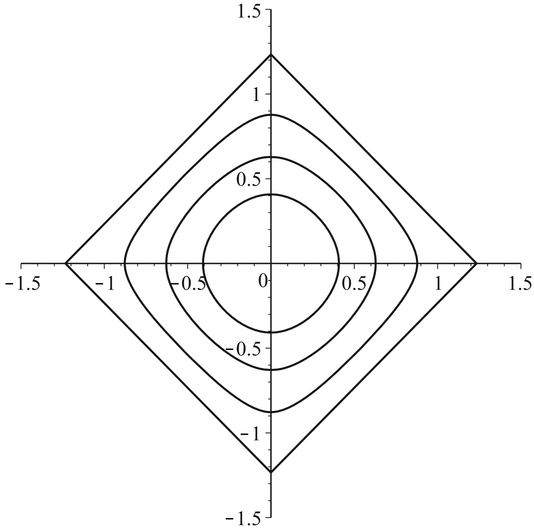


Fig. 3 Images of $\Lambda_{0,-1}[f_1](|z| = r)$ for $r = 0.4, 0.6, 0.8, 1$

For $\theta \in (0, \pi)$, write $(\Psi_c * f_1)(e^{i\theta}) = x + iy$. Then for $\theta \in (0, \pi/2]$,

$$y = -x + \frac{\pi}{4}\frac{1}{1+c}\left(2c + \frac{\pi}{2}\right), \quad x \in \frac{\pi}{4}\frac{1}{1+c}\left(c, c + \frac{\pi}{2}\right)$$

and for $\theta \in [\pi/2, \pi)$,

$$y = x + \frac{\pi}{4}\frac{1}{1+c}\left(2c + \frac{\pi}{2}\right), \quad x \in \frac{\pi}{4}\frac{1}{1+c}\left(-c - \frac{\pi}{2}, -c\right).$$

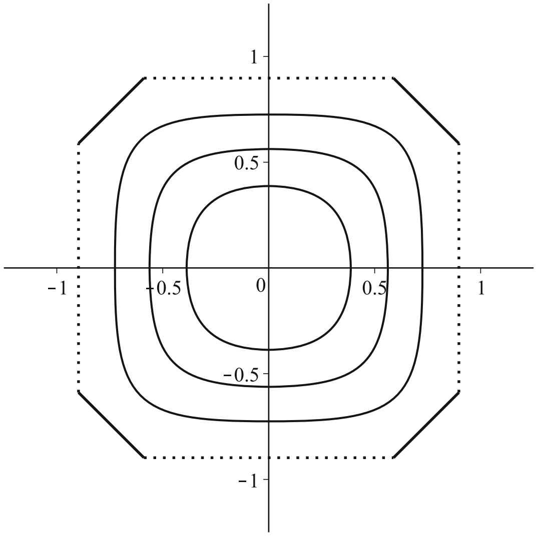


Fig. 4 Images of $(\Psi_3 * f_1)(|z| = r)$ for $r = 0.4, 0.6, 0.8$, $(\Psi_3 * f_1)(\partial\mathbb{D}\backslash\{\pm1, \pm i\})$ and the cluster set of $\Psi_3 * f_1$ at $\pm1, \pm i$ (*dotted*)

By symmetry, if $\theta \in (\pi, 2\pi)$, one has the reflection in the real axis of the above line segments. Since the real and imaginary parts of f_1 and F_1 are bounded on $\mathbb{D}$, from the convex combination representation of $\Psi_c * f_1$, we easily see the same is true of $\Psi_c * f_1$. Therefore, using the results of [5], the cluster sets of $\Psi_c * f_1$ at $\pm1, \pm i$ are the line segments connecting the four line segments given by the formulas in Eq. (3.8) and $\Psi_c * f_1$ maps $\mathbb{D}$ onto a convex octagon for each $c > 0$. See Fig. 4. □

The next corollary of Theorem 3.8 involves the convolution of the half-strip mapping Ψ_c with another half-strip mapping producing a family of unbounded convex polygonal mappings contained in a half-strip.

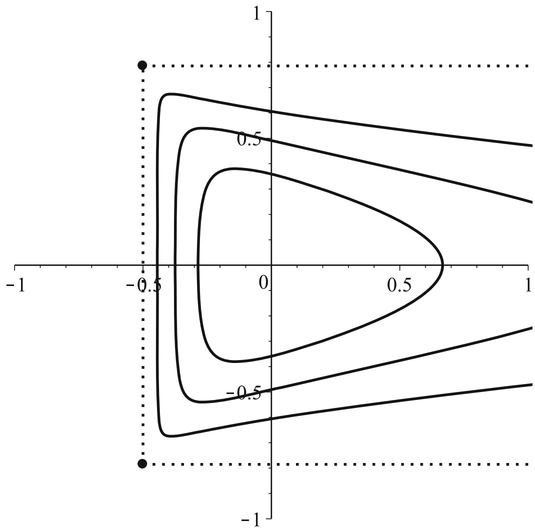
Corollary 3.10 *Let* $f_2 = h_2 + \overline{g_2} \in \mathcal{S}_H^0$ *be the mapping such that* $h_2(z) + g_2(z) = z/(1-z)$ *and* $g_2'(z)/h_2'(z) = z$. *Then for each* $c > 0$, $\Psi_c * f_2$ *is a convex mapping of* $\mathbb{D}$ *contained in the half-strip* $\{z \in \mathbb{C} : \operatorname{Re} z > -(\pi^2 + 4c)/(8(1+c)), |\operatorname{Im} z| < \pi(2+c)/(4(1+c))\}$.

Proof A calculation shows

$$f_2(z) = \frac{1}{2}\frac{z}{1-z} + \frac{1}{4}\log\frac{1+z}{1-z} + \overline{\frac{1}{2}\frac{z}{1-z} - \frac{1}{4}\log\frac{1+z}{1-z}}. \tag{3.9}$$

It is not hard to see that f_2 maps $\mathbb{D}$ onto the half-strip given by $\{z : \operatorname{Re} z > -1/2, |\operatorname{Im} z| < \pi/4\}$. See Fig. 5. By Eq. (3.9), $(h_2 - g_2)(z) = 1/2\log((1+z)/(1-z))$ which is a convex mapping. Hence, by Theorem 3.4, $\Psi_c * f_2$ is univalent and convex in the direction of the real axis for each $c > 0$.

Fig. 5 Images of $f_2(|z| = r)$ for $r = 0.4, 0.6, 0.8$, $f_2(\partial\mathbb{D}\backslash\{\pm 1\})$ and the cluster sets of f_2 and ± 1 (*dotted*)



In order to use Theorem 3.8, we again need to establish the mapping properties of $F_2 = \Lambda_{0,-1}[f_2]$. By Theorem A, we know $F_2 \in \mathcal{K}_H$. In fact, F_2 maps $\mathbb{D}$ onto a triangle. A calculation shows

$$F_2(z) = \frac{1}{2}\mathrm{Re}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(-z)\right) + i\,\mathrm{Im}\left(-\log(1-z)\right). \tag{3.10}$$

Through a Fourier series calculation, if $\theta \in (0, 2\pi)$,

$$\begin{aligned} F_2(e^{i\theta}) &= \sum_{n=0}^{\infty} \frac{\cos((2n+1)\theta)}{(2n+1)^2} - i\arg(1-e^{i\theta}) \\ &= \begin{cases} \dfrac{\pi}{4}\left(-\theta + \dfrac{\pi}{2}\right) + i\dfrac{-\theta+\pi}{2} & \text{if } \theta \in (0,\pi], \\ \dfrac{\pi}{4}\left(\theta - \dfrac{3\pi}{2}\right) + i\dfrac{-\theta+\pi}{2} & \text{if } \theta \in [\pi, 2\pi). \end{cases} \end{aligned} \tag{3.11}$$

From Eq. (3.11), F_2 maps the arc $\{e^{i\theta} : \theta \in (0,\pi]\}$ onto the line segment from $\pi^2/8 + i\pi/2$ (but not including) to $-\pi^2/8$ and reflecting this line segment in the real axis gives the image of the arc $\{e^{i\theta} : \theta \in [\pi, 2\pi)\}$ under F_2. Lastly, using that the real part of F_2 is bounded on $\mathbb{D}$ and results from [5], the cluster set of F_2 at 1 is the line segment joining $\pi^2/8 + i\pi/2$ and $\pi^2/8 - i\pi/2$. In other words, $F_2(\mathbb{D})$ is a triangle. See Fig. 6.

Since $h_2 \pm g_2$ are convex, by Theorem 3.2, $\Psi_c * f_2$ is locally univalent for each $c > 0$. Modifying the calculations of Example 1 in [8] using I_c in lieu of simply $I_1 = \ell_0$ shows $I_c * f_2$ is starlike. Hence, $\Psi_c * f_2$ is convex by Theorem 3.6. We may now use Theorem 2.4 in [1] in concert with the convex combination relationship of

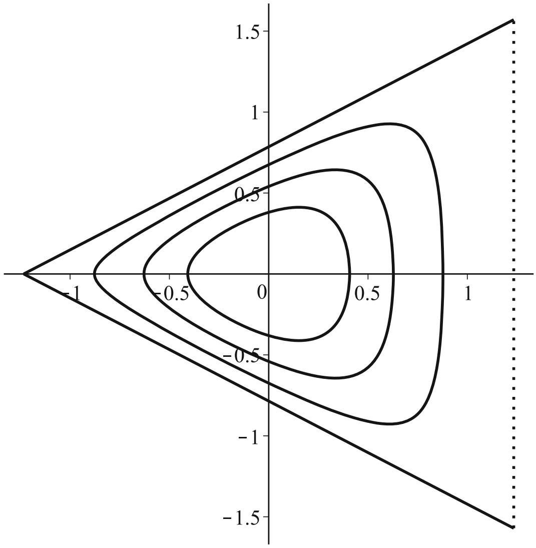


Fig. 6 Images of $\Lambda_{0,-1}[f_2](|z|=r)$ for $r=0.4, 0.6, 0.8$, $\Lambda_{0,-1}[f_2](\partial\mathbb{D}\backslash\{1\})$ and the cluster set of $\Lambda_{0,-1}[f_2]$ at 1 (*dotted*)

Theorem 3.8 to fully describe the image of $\mathbb{D}$ under $\Psi_c * f_2$. From the formulas in Eq. (3.11), if $\theta \in (0,\pi)\cup(\pi,2\pi)$

$$(\Psi_c * f_2)(e^{i\theta}) = \begin{cases} \dfrac{1}{1+c}\left(\dfrac{\pi}{4}\left(-\theta+\dfrac{\pi}{2}\right)-\dfrac{c}{2}+i\left(\dfrac{-\theta+\pi}{2}+\dfrac{c\pi}{4}\right)\right) & \text{if } \theta\in(0,\pi), \\ \dfrac{1}{1+c}\left(\dfrac{\pi}{4}\left(\theta-\dfrac{3\pi}{2}\right)-\dfrac{c}{2}+i\left(\dfrac{-\theta+\pi}{2}-\dfrac{c\pi}{4}\right)\right) & \text{if } \theta\in(\pi,2\pi). \end{cases} \tag{3.12}$$

For $\theta \in (0,\pi)\cup(\pi,2\pi)$, write $(\Psi_c * f_2)(e^{i\theta}) = x+iy$. Then

$$x \in \frac{1}{1+c}\left(-\frac{\pi^2}{8}-\frac{c}{2}, \frac{\pi^2}{8}-\frac{c}{2}\right).$$

If $\theta\in(0,\pi)$,

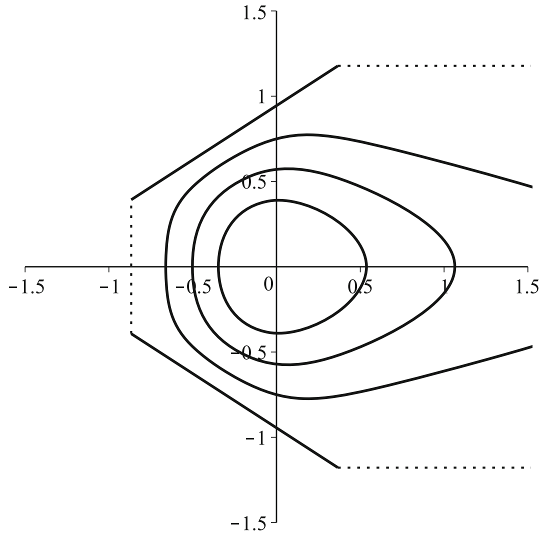
$$y = \frac{2}{\pi}x + \frac{c}{\pi(1+c)} + \frac{\pi}{4} \tag{3.13}$$

and if $\theta\in(\pi,2\pi)$,

$$y = -\frac{2}{\pi}x - \frac{c}{\pi(1+c)} - \frac{\pi}{4}. \tag{3.14}$$

Clearly, $(\Psi_c * f_2)(\mathbb{D})$, while convex, is neither a strip nor a half-plane. By Theorem 2.4 in [1], only one point on $\partial\mathbb{D}$ corresponds to infinity and we observe the real part of $\Psi_c * f_2$ is bounded below on $\mathbb{D}$ because the real parts of f_2 and F_2 are as well.

Fig. 7 Images of $(\Psi_1 * f_2)(|z| = r)$ for $r = 0.4, 0.6, 0.8$, $(\Psi_1 * f_2)(\partial\mathbb{D}\setminus \pm 1\})$ and the cluster set of $\Psi_1 * f_2$ at ± 1 (*dotted*)



Hence, the cluster set of $\Psi_c * f_2$ at -1 is the vertical line segment connecting the left endpoints of line segment described in Eq. (3.13) and its reflection in the real axis given by Eq. (3.14) and the cluster set of $\Psi_c * f_2$ at 1 are two parallel rays emanating from the right endpoint of the line segment described in Eq. (3.13) and its reflection in the real axis. See Fig. 7. □

In our next corollary to Theorem 3.8, as in the previous one, the ranges of the convolutions under consideration will be unbounded. In the proof of Corollary 3.10, we were able to use Theorem 2.4 in [1] because we knew $\Psi_c * f_2$ was convex by Theorem 3.6; however, in this next corollary, we cannot exploit Theorem 3.6 in the same way. Further, Theorem 3.4 does not apply in the next corollary either. So we will directly show the convolutions are convex mappings.

Corollary 3.11 *Let $f_3 = h_3 + \overline{g_3} \in \mathcal{S}_H^0$ be the mapping such that*

$$h_3(z) - g_3(z) = \frac{1}{2}\log\frac{1+z}{1-z}$$

and $g_3'(z)/h_3'(z) = z^2$. Then for each $c > 0$, $\Psi_c * f_3$ maps $\mathbb{D}$ onto the horizontal strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi/4\}$.

Proof A calculation gives

$$f_3(z) = \frac{1}{2}\frac{z}{1-z^2} + \frac{1}{4}\log\frac{1+z}{1-z} + \overline{\frac{1}{2}\frac{z}{1-z^2} - \frac{1}{4}\log\frac{1+z}{1-z}}. \tag{3.15}$$

It can be shown f_3 maps $\mathbb{D}$ onto the strip $\{z : |\mathrm{Im}\, z| < \pi/4\}$. See for example [9]. Once again this means $\Lambda_{0,-1}[f_3] \in \mathcal{K}_H$. A calculation shows

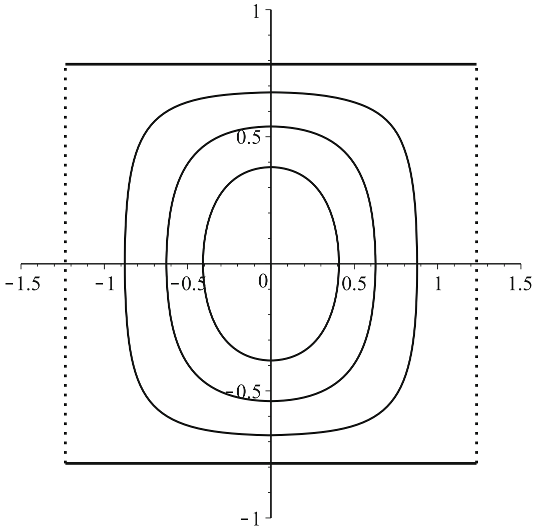


Fig. 8 Images of $\Lambda_{0,-1}[f_3](|z| = r)$ for $r = 0.4, 0.6, 0.8$, $\Lambda_{0,-1}[f_3](\partial\mathbb{D}\backslash\{\pm 1\})$ and the cluster set of $\Lambda_{0,-1}[f_3]$ at ± 1 (*dotted*)

$$\Lambda_{0,-1}[f_3](z) = \frac{1}{2}\text{Re}\,(\text{Li}_2(z) - \text{Li}_2(-z)) + \frac{i}{2}\text{Im}\,\log\frac{1+z}{1-z}. \tag{3.16}$$

Using a similar approach as in Corollaries 3.9 and 3.10, if we set $\Lambda_{0,-1}[f_3] = F_3$, for $\theta \in (0, \pi) \cup (\pi, 2\pi)$,

$$F_3(e^{i\theta}) = \begin{cases} \frac{\pi}{4}\left(-\theta + \frac{\pi}{2}\right) + i\frac{\pi}{4} & \text{if } \theta \in (0, \pi), \\ \frac{\pi}{4}\left(\theta - \frac{3\pi}{2}\right) - i\frac{\pi}{4} & \text{if } \theta \in (\pi, 2\pi). \end{cases} \tag{3.17}$$

which describes two horizontal line segments. Since the real part of F_3 is bounded on $\mathbb{D}$, the cluster set of F_3 at ± 1 are the vertical line segments connecting those given in Eq. (3.17) and $F_3(\mathbb{D})$ is a rectangle. See Fig. 8.

Observe $h_3 - g_3$ is convex and $h_3 + g_3$ is starlike. Thus, $\Psi_c * f_3$ is locally univalent by Theorem 3.2. We will directly show $\Psi_c * f_3$ is a horizontal strip mapping and hence convex. Using the formulas in Eq. (3.17), for $\theta \in (0, \pi) \cup (\pi, 2\pi)$

$$(\Psi_c * f_3)(e^{i\theta}) = \begin{cases} \frac{1}{1+c}\left(-\frac{\pi}{4}\theta + \frac{\pi^2}{8}\right) + i\frac{\pi}{4} & \text{if } \theta \in (0, \pi), \\ \frac{1}{1+c}\left(\frac{\pi}{4}\theta - \frac{3\pi^2}{8}\right) - i\frac{\pi}{4} & \text{if } \theta \in (\pi, 2\pi). \end{cases} \tag{3.18}$$

Thus, $\Psi_c * f_3$ maps $\partial\mathbb{D}\backslash\{\pm 1\}$ to the horizontal line segment $-\pi^2/(8(1+c)) + i\pi/4$ to $\pi^2/(8(1+c)) + i\pi/4$ and its reflection in the real axis. Furthermore,

$$\text{Im}(\Psi_c * f_3)(z) = \frac{1}{2}\text{Im}\,\log\frac{1+z}{1-z}.$$

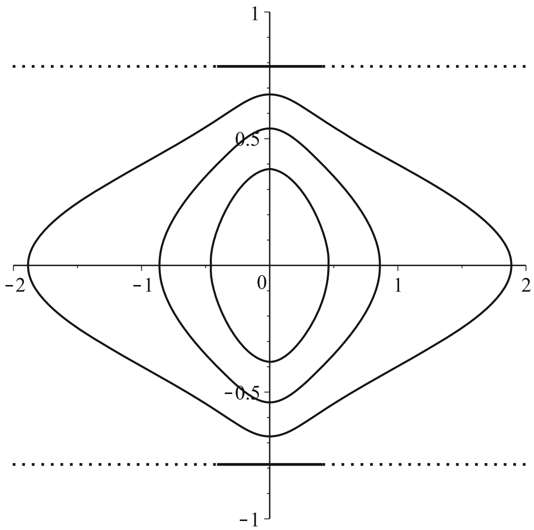


Fig. 9 Images of $(\Psi_2 * f_3)(|z| = r)$ for $r = 0.4, 0.6, 0.8$, $(\Psi_2 * f_3)(\partial\mathbb{D}\backslash\{\pm 1\})$ and the cluster set of $\Psi_2 * f_3$ at ± 1 (*dotted*)

Thus, by the work of Hengartner and Schober [10], the image of $\mathbb{D}$ under $\Psi_c * f_3$ is convex and contained in the horizontal strip $\{z : |\text{Im}\, z| < \pi/4\}$. Since the real part of $\Psi_c * f_3$ is unbounded in both directions, it must map onto the full strip. See Fig. 9.

□

In Theorem 3.2, for $f = h + \overline{g} \in \mathcal{S}_H$, it was enough for $h \pm g$ to be starlike for $\Psi_c * f$ to be locally univalent. These conditions were met for f_1, f_2, and f_3 in Corollaries 3.9, 3.10, and 3.11, respectively, and in each case, we remark $\Psi_c * f_j$, $j = 1, 2, 3$ was in fact convex.

4 Generalized Harmonic Bernardi Integral Operator

The generalized harmonic Bernardi integral operator $\Lambda_{\lambda,\alpha}[f] : \mathbb{D} \to \mathbb{C}$, $\text{Re}\,\lambda \geq 0$, $\alpha \in \overline{\mathbb{D}}$, and $f \in \mathcal{S}_H$ was defined in Eq. (1.4) as

$$\Lambda_{\lambda,\alpha}[f](z) = \frac{\lambda+1}{z^\lambda}\int_0^z \zeta^{\lambda-1}h(\zeta)\,d\zeta + \alpha\overline{\frac{\lambda+1}{z^\lambda}\int_0^z \zeta^{\lambda-1}g(\zeta)\,d\zeta}.$$

Recall the half-strip mappings $\Psi_c = \Lambda_{0,-1}[I_c]$, and in general, from Theorem A, $\Lambda_{0,-1}[\mathcal{S}_H^*] \subseteq \mathcal{K}_H$. A "positive" Alexander operator equivalent to $\Lambda_{0,1}$ was studied in [21] and in [20], the authors introduced a harmonic Libera operator corresponding to $\Lambda_{1,1}$. These recent works established invariance results of certain classes of harmonic mappings under $\Lambda_{0,1}$ and $\Lambda_{1,1}$. In this section, we explore some invariance properties of the generalized harmonic Bernardi integral operator $\Lambda_{\lambda,\alpha}$ which will encompass the results for the positive Alexander and the Libera operator.

The crux of the arguments in this section rely on identifying an additional convolution representation for $\Lambda_{\lambda,\alpha}$ as was done in Eq. (3.3) in the proof of Theorem 3.6. Thus, we begin with this lemma.

Lemma 4.1 *Define for* $\operatorname{Re}\lambda \geq 0$, $\Phi_\lambda : \mathbb{D} \to \mathbb{C}$ *as*

$$\Phi_\lambda(z) = z + \sum_{n=2}^{\infty} \left(\frac{\lambda+1}{\lambda+n}\right) z^n. \tag{4.1}$$

If $f \in \mathcal{S}_H$ *and* $\Lambda_{\lambda,\alpha}$ *is as defined in Eq.* (1.4),

$$\Lambda_{\lambda,\alpha}[f](z) = (\Phi_\lambda * h)(z) + \alpha\overline{(\Phi_\lambda * g)(z)}. \tag{4.2}$$

Proof The lemma follows from the fact that if $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathcal{H}_0(\mathbb{D})$ then

$$(\Phi_\lambda * f)(z) = \sum_{n=1}^{\infty} \left(\frac{\lambda+1}{\lambda+n}\right) a_n z^n = \frac{\lambda+1}{z^\lambda} \int_0^z \zeta^{\lambda-1} f(\zeta)\, d\zeta. \tag{4.3}$$

□

Our first geometric result for the generalized harmonic Bernardi integral operator is the following corollary.

Corollary 4.2 *If* $f \in \mathcal{K}_H$, *then* $\Lambda_{\lambda,\alpha}[f] \in \mathcal{C}_H$.

Proof Let Φ_λ be as defined in Eq. (4.1). For $f \in \mathcal{H}_0(\mathbb{D})$, it is known [16, p. 67] $\Phi_\lambda * f$ is convex, starlike, or close-to-convex whenever f is, respectively. As $I(z) = z/(1-z)$ is the identity under the Hadamard product for $f \in \mathcal{H}_0(\mathbb{D})$, it is clear $\Phi_\lambda \in \mathcal{K}$ for each λ, $\operatorname{Re}\lambda \geq 0$. By Lemma 4.1 and the convolution theorem in [6], this corollary is now evident. □

In order to utilize further the convolution representation form of $\Lambda_{\lambda,\alpha}[f]$ given in Eq. (4.2) of Lemma 4.1, we first prove following more general result for harmonic convolutions.

Theorem 4.3 *Let* $f = h + \overline{g}$ *be locally univalent and suppose* $h + \varepsilon g$ *is convex for some* $|\varepsilon| = 1$. *If* $\varphi \in \mathcal{K}$ *and* $|\alpha| = 1$, $(\varphi + \alpha\overline{\varphi}) * f$ *is close-to-convex. In particular,* $(\varphi + \alpha\overline{\varphi}) * f$ *is convex in the direction of* $(\arg\varepsilon - \arg\alpha)/2 + \pi/2$. *Thus,* $(\varphi + \varepsilon\overline{\varphi}) * f$ *is convex in the direction of the imaginary axis and* $(\varphi - \varepsilon\overline{\varphi}) * f$ *is convex in the direction of the real axis.*

Our proof uses the following lemma from [23].

Lemma A *Let* φ *be convex and* g *be starlike in* $\mathbb{D}$. *Then for each function* P *analytic on* $\mathbb{D}$ *with* $\operatorname{Re} P(z) > 0$, $z \in \mathbb{D}$,

$$\operatorname{Re} \frac{(\varphi * Pg)(z)}{(\varphi * g)(z)} > 0,\ z \in \mathbb{D}. \tag{4.4}$$

Proof of Theorem 4.3 Write $(\varphi + \alpha\overline{\varphi}) * f = H + \overline{G}$. We first show $H + \overline{G}$ is locally univalent or that $|G'(z)/H'(z)| < 1$, $z \in \mathbb{D}$. Consider for $z \in \mathbb{D}$

$$\operatorname{Re} \frac{1 - \dfrac{\varepsilon(\varphi * g)'(z)}{(\varphi * h)'(z)}}{1 + \dfrac{\varepsilon(\varphi * g)'(z)}{(\varphi * h)'(z)}} = \operatorname{Re} \frac{\varphi(z) * z(h - \varepsilon g)'(z)}{\varphi(z) * z(h + \varepsilon g)'(z)}$$
$$= \operatorname{Re} \frac{\varphi(z) * \left(z(h + \varepsilon g)'(z) \dfrac{z(h - \varepsilon g)'(z)}{z(h + \varepsilon g)'(z)} \right)}{\varphi(z) * z(h + \varepsilon g)'(z)}$$
$$= \operatorname{Re} \frac{\varphi(z) * \left(z(h + \varepsilon g)'(z) \dfrac{1 - \varepsilon g'(z)/h'(z)}{1 + \varepsilon g'(z)/h'(z)} \right)}{\varphi(z) * z(h + \varepsilon g)'(z)}. \tag{4.5}$$

Since f is locally univalent, $|g'(z)/h'(z)| < 1$, $z \in \mathbb{D}$. Thus,

$$P(z) = \frac{1 - \varepsilon g'(z)/h'(z)}{1 + \varepsilon g'(z)/h'(z)}$$

has positive real part for $z \in \mathbb{D}$. Furthermore, since $h + \varepsilon g$ is convex, $z(h + \varepsilon g)'(z)$ is starlike. Hence, Lemma A shows the expression in (4.5) is positive for $z \in \mathbb{D}$ and so $H + \overline{G}$ is locally univalent.

For $\varepsilon_1 = \varepsilon/\alpha$, $H + \varepsilon_1 G = \varphi * (h + \varepsilon g)$ is convex which implies $(\varphi + \alpha\overline{\varphi}) * f$ is univalent close-to-convex (see [6]). Lastly, Theorem B completes the direction convexity statements. □

As Φ_λ in Eq. (4.1) is convex, we have the immediate corollary.

Corollary 4.4 *Let $f = h + \overline{g}$ be locally univalent. Suppose $h + \varepsilon g$ is convex for some $|\varepsilon| = 1$. Then for $|\alpha| = 1$, $\Lambda_{\lambda,\alpha}[f]$ is close-to-convex. In particular, $\Lambda_{\lambda,\alpha}[f]$ is convex in the direction of* $(\arg \varepsilon - \arg \alpha)/2 + \pi/2$. *Thus, if $\varepsilon = \alpha$, $\Lambda_{\lambda,\alpha}[f]$ is convex in the direction of the imaginary axis and if $\varepsilon = -\alpha$, $\Lambda_{\lambda,\alpha}[f]$ is convex in the direction of the real axis.*

Corollary 4.4 is a generalization of recent work in [22] where it was shown that if f is a harmonic mapping convex in the direction of φ, $\Lambda_{0,-1}[f]$ is convex in the direction of $\varphi + \pi/2$. That is, f and $\Lambda_{0,-1}[f]$ are convex in orthogonal directions.

To illustrate this corollary, consider $f_1 = h_1 + \overline{g_1}$ in Corollary 3.9 and $f_2 = h_2 + \overline{g_2}$ in Corollary 3.10 where we observed $h_j \pm g_j$, $j = 1, 2$, were convex mappings. By Corollary 4.4, for each $\operatorname{Re} \lambda \geq 0$ and $|\alpha| = 1$, $\Lambda_{\lambda,\alpha}[f_j]$, $j = 1, 2$ is convex in the direction of $-(\arg \alpha)/2 + \pi/2$ and $\pi - (\arg \alpha)/2$. In other words, for each $j = 1, 2$, $\Lambda_{\lambda,\alpha}[f_j]$ is a mapping convex in orthogonal directions. In particular, for $j = 1, 2$, $\Lambda_{\lambda,1}[f_j]$ and $\Lambda_{\lambda,-1}[f_j]$ are each convex in the direction of both the imaginary and real axes. For f_3 in Corollary 3.11, we note $h_3 + g_3$ is starlike and not convex but $h_3 - g_3$ is convex. Hence, for each $\operatorname{Re} \lambda \geq 0$ and $|\alpha| = 1$, $\Lambda_{\lambda,-1}[f_3]$ is convex in the direction of the imaginary axis and $\Lambda_{\lambda,1}[f_3]$ is convex in the direction of the real axis.

It is well known that the hypotheses on f of Theorem 4.3 imply f is close-to-convex [6, Theorem 5.15]. Thus, by Corollary 4.4, $\Lambda_{\lambda,\alpha}$ maps a subset of $\mathcal{C}_H$ into $\mathcal{C}_H$. We can see stronger invariance results for $\Lambda_{\lambda,\alpha}$ which generalize several results in [20] by introducing the idea of a stable harmonic family [11]. A harmonic function $f = h + \overline{g}$ is said to be in a stable family (e.g. stable close-to-convex) if $f_\alpha = h + \alpha\overline{g}$ is in the family (e.g. close-to-convex) for each $|\alpha| = 1$. Further, let $\mathcal{G} \subseteq \mathcal{S}$ and define the harmonic analogue of $\mathcal{G}$ to be the set $\mathcal{G}_H^0 \subseteq \mathcal{S}_H^0$ of functions $f = h + \overline{g}$ such that $h + \alpha g \in \mathcal{G}$ for each $|\alpha| = 1$. In this case, write $\mathcal{G} \triangleright \mathcal{G}_H^0$ and call $\mathcal{G}_H^0$ the harmonic analogue of $\mathcal{G}$. The following was proved in [20].

Theorem G *Let $\mathcal{G} \subseteq \mathcal{S}$ and $\mathcal{G} \triangleright \mathcal{G}_H^0$. Let $\mathcal{O} \subseteq \mathcal{H}_0(\mathbb{D})$ be such that $\mathcal{G}$ is closed under convolution with members of $\mathcal{O}$. Then $(\varphi + \overline{\varphi}) * f \in \mathcal{G}_H^0$ for all $\varphi \in \mathcal{O}$ and $f \in \mathcal{G}_H^0$.*

In [20], the authors proved $\Lambda_{0,1}[\mathcal{K}_H^0] \subseteq \mathcal{SC}_H^0$ and $\Lambda_{1,1}[\mathcal{K}_H^0] \subseteq \mathcal{SC}_H^0$ where $\mathcal{SC}_H^0$ is the normalized family of stable close-to-convex harmonic mappings. They also proved that $\Lambda_{0,1}$ and $\Lambda_{1,1}$ preserve $\mathcal{SK}_H^0$ and $\mathcal{SS}_H^{*0}$, the families of normalized harmonic stable convex and stable starlike mappings, respectively. Evidently, Corollaries 4.2 and 4.4 connect us to stable harmonic families. With this observation, the fact that whenever $\operatorname{Re}\lambda \geq 0$ and $f \in \mathcal{H}_0(\mathbb{D})$ is close-to-convex, starlike, or convex, so is $\Phi_\lambda * f$, respectively, where Φ_λ is as given in Eq. (4.1), and knowing $\mathcal{C} \triangleright \mathcal{SC}_H^0$, $\mathcal{S}^* \triangleright \mathcal{SS}_H^{*0}$, and $\mathcal{K} \triangleright \mathcal{SK}_H^0$, we generalize Theorems 4.2, 4.5, 4.6, and 4.9 from [20] using $\mathcal{O} = \{\Phi_\lambda(z) : \operatorname{Re}\lambda \geq 0\}$ in Theorem G and the linearity of the convolution.

Theorem 4.5 *For each $\operatorname{Re}\lambda \geq 0$ and $|\alpha| = 1$, $\Lambda_{\lambda,\alpha}$ preserves the families $\mathcal{SK}_H^0$, $\mathcal{SS}_H^{*0}$, and $\mathcal{SC}_H^0$. Moreover, $\Lambda_{\lambda,\alpha}[\mathcal{K}_H^0] \subseteq \mathcal{SC}_H^0$.*

We conclude this section with a weak subordination result for $\Lambda_{\lambda,1}[f]$ which again takes advantage of the convolution representation from Lemma 4.1.

Theorem 4.6 *If $f \in \mathcal{K}_H$ and $\lambda > 0$, $\lambda/(\lambda + 1)\Lambda_{\lambda,1}[f]$ is weakly subordinate to f.*

Proof In [18], it was shown whenever $F \in \mathcal{K}$ and $0 < \lambda_1 < \lambda_2$,

$$\frac{\lambda_1}{1+\lambda_1}\Phi_{\lambda_1} * F \prec \frac{\lambda_2}{1+\lambda_2}\Phi_{\lambda_2} * F \prec F \tag{4.6}$$

where Φ_λ is as given in Eq. (4.1). Taking $F(z) = z/(1-z)$, $\lambda/(1+\lambda)\Phi_\lambda \prec z/(1-z)$ for each $\lambda > 0$. Consequently, for each $\lambda > 0$,

$$\frac{\lambda}{1+\lambda}\Phi_\lambda(z) = \int_0^{2\pi} \frac{e^{-it}z}{1-e^{-it}z}\, d\mu_\lambda(t)$$

for some probability measure μ_λ. Hence, for $f = h + \overline{g} \in \mathcal{K}_H$ and $z \in \mathbb{D}$,

$$\frac{\lambda}{1+\lambda}\left(\left(\Phi_\lambda + \overline{\Phi_\lambda}\right) * f\right)(z) = \int_0^{2\pi} \left(h(e^{-it}z) + \overline{g(e^{-it}z)}\right) d\mu_\lambda(t) \in f(\mathbb{D}).$$

Therefore, $\lambda/(\lambda + 1)\Lambda_{\lambda,1}[f]$ is weakly subordinate to f. □

Relating this back to the half-plane mappings I_c in Eq. (1.2) and other mappings throughout the paper, for each positive c and λ, $\lambda/(1+\lambda)\Lambda_{\lambda,1}[I_c] \in \mathcal{C}_H$ is weakly subordinate to the half-plane mapping I_c; while for each $\lambda > 0$ and $j = 1, 2, 3$, $\lambda/(1+\lambda)\Lambda_{\lambda,1}[f_j] \in \mathcal{C}_H$ is weakly subordinate to the convex mapping f_j given in Corollaries 3.9, 3.10, and 3.11.

5 Some Final Remarks

In Theorem 3.2, we proved that for $f = h + \overline{g} \in \mathcal{S}_H$ it was sufficient for $h \pm g$ to be starlike for $\Psi_c * f$ to be locally univalent. These conditions were met for f_1, f_2, and f_3 in Corollaries 3.9, 3.10, and 3.11, respectively, and in each case, $\Psi_c * f_j$, $j = 1, 2, 3$ was in fact convex. This starlike condition for $h \pm g$ is unlike other results on local univalence thus far in the study of harmonic Hadamard products. When trying to determine if the convolution of $f = h + \overline{g}$ with Ψ_c is locally univalent, one may now ask whether there are any dilatations for which $h + g$ (or $h - g$) will be starlike when shearing with $h - g$ (or $h + g$) equal to a starlike mapping. Further, it would be of interest to understand how the starlikeness $h \pm g$ is playing into the geometry of $\Psi_c * f$.

We have numerical evidence that leads us to suspect that in addition to the starlikeness of $h \pm g$, the boundary behavior of f collapsing to point(s) might be the key to the convexity behavior seen in Corollaries 3.9, 3.10, and 3.11, the reason being that the normal to $\Lambda_{0,-1}[f]$ for any f and for $|z| = 1$ is given by f itself, provided f is continuous to $\partial\mathbb{D}$. Moreover, to underscore the necessity of starlikeness for the convolution to be a convex mapping, we consider $f_4 = h_4 + \overline{g_4} : \mathbb{D} \to \mathbb{C}$ that is a shear in the vertical direction of $I(z) = z/(1-z)$ like f_2 in Corollary 3.10 but $g_4'(z)/h_4'(z) = -z^2$. In this case, $\Psi_1 * f_4$ is not convex.

A calculation gives

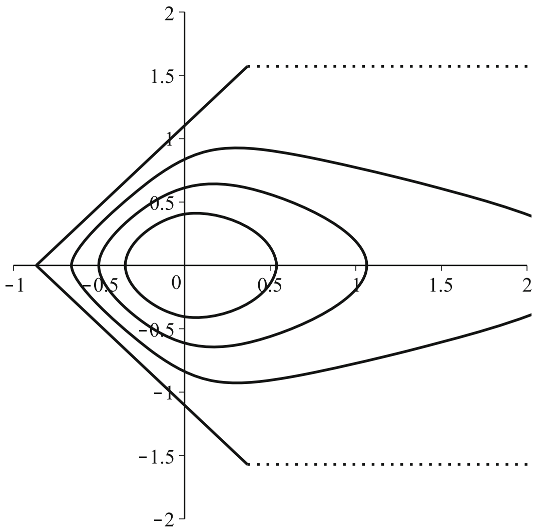
$$f_4(z) = \frac{1}{8}\left(\frac{4z}{1-z} + \log\frac{1+z}{1-z} + \frac{2z}{(1-z)^2} + \overline{\frac{4z}{1-z} - \log\frac{1+z}{1-z} - \frac{2z}{(1-z)^2}}\right). \tag{5.1}$$

Here $h_4 - g_4$ is not starlike. Rather, it is a two slit mapping taking $\mathbb{D}$ onto $\mathbb{C}\backslash\{x \pm i\pi/8 : x \le -1/4\}$. It can be shown that f_4 maps $\mathbb{D}$ onto the right half-plane $\{z : \mathrm{Re}\, z > -1/2\}$. Hence, by Theorem 3.6, $\Lambda_{0,-1}[f_4] \in \mathcal{K}_H$. A calculation shows

$$\Lambda_{0,-1}[f_4](z) = \mathrm{Re}\left(\frac{1}{4}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(-z)\right) + \frac{1}{2}\frac{z}{1-z}\right) - i\mathrm{Im}\left(\log(1-z)\right). \tag{5.2}$$

Set $\Lambda_{0,-1}[f_4] = F_4$. Through the same calculations as in Corollary 3.10, if $\theta \in (0, 2\pi)$,

Fig. 10 Images of $\Lambda_{0,-1}[f_4](|z| = r)$ for $r = 0.4, 0.6, 0.8$, $\Lambda_{0,-1}[f_4](\partial\mathbb{D}\backslash\{1\})$, and the cluster set of $\Lambda_{0,-1}[f_4]$ at 1 (*dotted*)



$$F_4(e^{i\theta}) = \begin{cases} \frac{\pi}{8}\left(-\theta + \frac{\pi}{2}\right) - \frac{1}{4} + i\frac{-\theta+\pi}{2} & \text{if } \theta \in (0, \pi], \\ \frac{\pi}{8}\left(\theta - \frac{3\pi}{2}\right) - \frac{1}{4} + i\frac{-\theta+\pi}{2} & \text{if } \theta \in [\pi, 2\pi). \end{cases} \tag{5.3}$$

From Eq. (5.3) and by similar arguments as in Corollary 3.10, we see $F_4(\mathbb{D}\backslash\{1\})$ gives line segments akin to F_1 but now the real part of F_4 is bounded below but not above on $\mathbb{D}$. Hence, the cluster set of F_4 at 1 is a pair of rays parallel to the real axis. See Fig. 10. If $\theta \in (0, \pi) \cup (\pi, 2\pi)$,

$$(\Psi_1 * f_4)(e^{i\theta}) = \begin{cases} \frac{1}{2}\left(\frac{\pi}{4}\left(-\theta + \frac{\pi}{2}\right) - \frac{3}{8} + i\left(-\frac{\theta}{4} + \frac{5\pi}{16}\right)\right) & \text{if } \theta \in (0, \pi), \\ \frac{1}{2}\left(\frac{\pi}{4}\left(\theta - \frac{3\pi}{2}\right) - \frac{3}{4} + i\left(-\frac{\theta}{4} + \frac{3\pi}{16}\right)\right) & \text{if } \theta \in (\pi, 2\pi). \end{cases} \tag{5.4}$$

See Fig. 11 for images of $\Psi_1 * f_4$ where we see the image is non-convex.

Further, by Corollary 4.4, we note that for each $\operatorname{Re}\lambda \geq 0$, $\Lambda_{\lambda,-1}[I_c]$ is convex in the direction of the real axis and $\Lambda_{\lambda,1}[I_c]$ is convex in the direction of the imaginary axis. The structure of $\Psi_c = \Lambda_{0,-1}[I_c]$ led to the convex combination decomposition of the convolution $\Psi_c * f$ in Theorem 3.8 that was instrumental in understanding the mapping properties of the convolution. Nonetheless, for any $\lambda \geq 0$ and $|\alpha| \leq 1$ we also have the more general convex combination decomposition

$$(\Lambda_{\lambda,\alpha}[I_c] * f)(z) = \frac{c}{1+c}\left[(\lambda+1)(h(z) - \alpha\overline{g(z)}) - \lambda\Lambda_{\lambda,-\alpha}[f](z)\right] + \frac{1}{1+c}\Lambda_{\lambda,\alpha}[f](z).$$

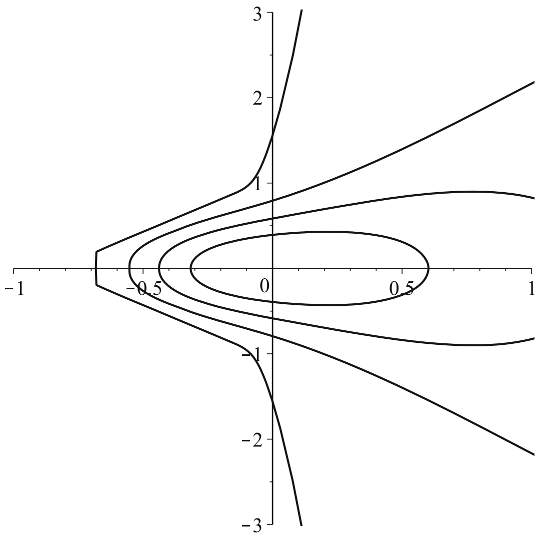


Fig. 11 Images of $(\Psi_1 * f_4)(|z| = r)$ for $r = 0.4, 0.6, 0.8, 0.999$

In terms of the weak subordination in Theorem 4.6, we suspect there are other results, including possibly weak subordination chains (see [19]), attainable from the operator $\Lambda_{\lambda,\alpha}$ because the analytic Bernardi integral operator produces convex subordination chains as described in Eq. (4.6). See [18] for more details and also notice $\Phi_\lambda(z) \to z/(1-z)$ as $\lambda > 0$ tends to infinity. As an example, through a similar change of variables as was used to show Ψ_c was a half-strip mapping, one can show $\partial\Lambda_{0,1}[I_c](\mathbb{D}\backslash\{1\})$ in the uv-plane is given by the curve $u = -1/(1+c)(2\ln 2c - \ln((1+c)^2 v^2 + c^2))$ and $\Lambda_{0,1}[I_c](\mathbb{D})$ is the region to the right of this bounding curve. Thus, $\Lambda_{0,1}[I_c]$ is weakly subordinate to $2I_c$ and we have also shown in Theorem 3.1, $(1/c)\Lambda_{0,-1}[I_c]$ is weakly subordinate to I_c. Compare these with Theorem 4.6.

Lastly, we note that the structure of the half-plane mapping I_c can be expanded as was done in [17] to form $T_c[f] : \mathbb{D} \to \mathbb{C}$ where

$$T_c[f](z) = \frac{f(z) + czf'(z)}{1+c} + \overline{\frac{f(z) - czf'(z)}{1+c}}, \quad c > 0$$

for $f \in \mathcal{K}$ or f in the subclass of $\mathcal{K}$ of direction convexity preserving functions. Many of the geometric results of Sect. 4 apply to $\Lambda_{\lambda,\alpha}[T_c(f)]$. For example, if we write $T_c(f) = h_c + \overline{g_c}$, then $h_c + g_c = 2/(1+c)f$ which maps $\mathbb{D}$ onto a convex domain. Hence, by Corollary 4.4, $\Lambda_{\lambda,\alpha}[T_c(f)]$ is (stable) close-to-convex for $|\alpha| = 1$. In particular, when $f \in \mathcal{K}$, $T_c(f)$ is convex in the direction of the imaginary axis [17] and so is $\Lambda_{\lambda,1}[T_c(f)]$.

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