Subordinate Solutions of a Differential Equation *

Stacey Muir †

Abstract

In 2003, Ruscheweyh and Suffridge settled a conjecture of Pólya and Schoenberg on subordination of the de la Vallée Poussin means of an analytic function by defining a continuous extension of the de la Vallée Poussin means using a differential equation. We extend this differential equation to a more general setting and observe that a similar subordination result with convex functions holds. Through an integral operator of Bernardi, particular convex subordination chains are constructed with specified limiting functions. Lastly, we show the importance of convexity by producing an example of a family of starlike solutions that fails to be a subordination chain.

1 Introduction

The de la Vallée Poussin means are defined by

$$v_n(f,t) = \frac{1}{2\pi}\int_0^{2\pi} \omega_n(t-\tau)f(\tau)\,d\tau, \qquad n \in \mathbb{N}$$

where f is a periodic real-valued function and

$$\omega_n(t) = \frac{2^n(n!)^2}{(2n)!}(1+\cos t)^n = \frac{1}{\binom{2n}{n}}\sum_{k=-n}^{n}\binom{2n}{n+k}e^{ikt}$$

are the de la Vallée Poussin kernels. We cast these means in terms of complex-valued analytic functions using the Hadamard product. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in the unit disk Δ. Then the

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†Department of Mathematics, University of Scranton, Scranton, PA 18510 U.S.A.

function $f * g$ given by $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ is called the Hadamard product of f and g. Define

$$V_n(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} z^k, \qquad n \in \mathbb{N}, z \in \Delta.$$

Let $\mathcal{H}_0(\Delta) = \{f : f \text{ is analytic in } \Delta, f(0) = 0\}$. For $f \in \mathcal{H}_0(\Delta)$, $V_n * f$ is the n^{th} de la Vallée Poussin mean of f. The function $V_n * f$ is a polynomial of degree n that approximates f in the sense that as n tends to infinity, $V_n * f$ tends to f locally uniformly.

Let

$$S = \{f \in \mathcal{H}_0(\Delta) : f'(0) = 1,\ f \text{ is univalent in } \Delta\}$$

be the family of normalized univalent analytic functions and let

$$K = \{f \in S :\ f(\Delta) \text{ is convex}\},$$

$$S^* = \{f \in S : f(\Delta) \text{ is starlike with respect to the origin}\},$$

and

$$C = \{f \in S : f(\Delta) \text{ is close-to-convex}\}$$

be the family of normalized convex, starlike, and close-to-convex univalent analytic functions, respectively. We will say a function f is convex, starlike, or close-to-convex if $f(z)/f'(0)$ defines a function in K, S^*, or C, respectively. For f and g in $\mathcal{H}_0(\Delta)$, we say f is subordinate to g, written $f \prec g$, if there exists a function $\omega \in \mathcal{H}_0(\Delta)$, $|\omega(z)| < 1$ for $z \in \Delta$, such that $f(z) = g(\omega(z))$. If g is univalent and $f(\Delta) \subset g(\Delta)$, then $f \prec g$.

In 1958, Pólya and Schoenberg [4] examined several shape-preserving properties of the de la Vallée Poussin means and conjectured if $f \in K$, the subordination chain

$$V_1 * f \prec V_2 * f \prec \cdots \prec f$$

exists. In 2003, Ruscheweyh and Suffridge [6] proved that the functions V_n satisfy the differential equation

$$zV_\lambda'(z) + \lambda \frac{1-z}{1+z} V_\lambda(z) = \lambda \frac{z}{1+z}, \qquad V_\lambda(0) = 0 \tag{1}$$

when $\lambda = n$. Additionally, this differential equation has analytic solutions whenever $\lambda > 0$. This led to the following result of Ruscheweyh and Suffridge [6] which settled Pólya and Schoenberg's above conjecture.

Theorem 1 (Ruscheweyh, Suffridge)**.** *For $f \in K$, we have $V_\lambda * f \in K$, for all $\lambda > 0$. Furthermore,*

$$V_{\lambda_1} * f \prec V_{\lambda_2} * f \prec f, \qquad 0 < \lambda_1 < \lambda_2.$$

The structure of the differential equation (1) is critical to the proof of the previous theorem and motivates this paper. We will explore mapping properties of solutions of

$$zF_\lambda'(z) + \lambda\psi(z)F_\lambda(z) = \lambda\phi(z), \qquad F_\lambda(0) = 0,\, \lambda > 0 \tag{2}$$

where ψ is an analytic function in Δ with positive real part, $\psi(0) = 1$, and $\phi \in \mathcal{H}_0(\Delta)$ with $\phi'(0) = 1$. In the following sections, we extend a subordination result of Ruscheweyh and Suffridge for convex functions using the differential equation (2). Through an integral operator of Bernardi, convex subordination chains are constructed with specified limiting functions. Lastly, we show the importance of convexity by producing an interesting example of a family of starlike solutions that fails to be a subordination chain.

2 Subordination Chains

Our main result on the subordination of families of solutions of the differential equation (2) is given in the following theorem.

Theorem 2. *If F_λ is a convex solution of the differential equation* (2) *for each $\lambda > 0$, then $F_{\lambda_1} \prec F_{\lambda_2}$ whenever $0 < \lambda_1 < \lambda_2$.*

To prove this, we will use the following result of Pommerenke [5].

Theorem 3 (Pommerenke)**.** *Let $a > b$ and $f : \Delta \times [a, b] \to \mathbb{C}$. Assume the following.*

1. *$f(\cdot, t) \in \mathcal{H}_0(\Delta)$ for each $t \in [a, b]$.*
2. *$f(\cdot, a)$ is univalent in Δ.*
3. $\left.\dfrac{\partial f(z,t)}{\partial z}\right|_{z=0} > 0$ *for each $t \in [a, b]$.*
4. *$f(z, \cdot) \in C^1[a, b]$ for each $z \in \Delta$.*

If $h : \Delta \times [a, b] \to \mathbb{C}$ defined by

$$\frac{\partial f(z,t)}{\partial t} = h(z,t) z \frac{\partial f(z,t)}{\partial z}$$

satisfies the condition $\operatorname{Re} h(z,t) > 0$ *for all* $z \in \Delta$ *and* $t \in [a, b]$*, then* $f(z,t)$ *is a univalent subordination chain. That is,* $f(\cdot, t)$ *is univalent in* Δ *for each fixed* $t \in [a, b]$ *and* $f(\cdot, s) \prec f(\cdot, t)$ *whenever* $a \le s < t \le b$*.*

The proof of Theorem 2 generalizes the proof Ruscheweyh and Suffridge [6] used when proving $\{V_\lambda\}_{\lambda > 0}$ is a subordination chain.

Proof of Theorem 2. Let λ_1 and λ_2 be fixed with $0 < \lambda_1 < \lambda_2$. Let $t \in [0, 1]$ and define $f(z,t) = tF_{\lambda_2}(z) + (1-t)F_{\lambda_1}(z)$. Then $f(z,t)$ is analytic in z for each $t \in [0,1]$ and $f(z,0) = F_{\lambda_1}(z)$ is univalent. Also, $f(z, \cdot) \in C^1[0,1]$. Since $F'_\lambda(0) = \lambda/(1+\lambda)$, we have

$$\left.\frac{\partial f(z,t)}{\partial z}\right|_{z=0} = t\frac{\lambda_2}{1+\lambda_2} + (1-t)\frac{\lambda_1}{1+\lambda_1} > 0.$$

We need to show

$$h(z,t) = \frac{\dfrac{\partial f(z,t)}{\partial t}}{z\dfrac{\partial f(z,t)}{\partial z}} = \frac{F_{\lambda_2}(z) - F_{\lambda_1}(z)}{tzF'_{\lambda_2}(z) + (1-t)zF'_{\lambda_1}(z)}$$

has positive real part, or equivalently $1/h(z,t)$ has positive real part, for $z \in \Delta$ and $t \in [0,1]$. Using the fact that $F_{\lambda_i}, i = 1, 2$, is a solution of $zF'_{\lambda_i}(z) + \lambda_i \psi(z) F_{\lambda_i} = \lambda_i \phi(z)$, we have

$$\frac{\lambda_2 zF'_{\lambda_1}(z) - \lambda_1 zF'_{\lambda_2}(z)}{F_{\lambda_2}(z) - F_{\lambda_1}(z)} = \lambda_1 \lambda_2 \psi(z). \tag{3}$$

We will show

$$\operatorname{Re}\left(\frac{zF'_{\lambda_2}(z)}{F_{\lambda_2}(z) - F_{\lambda_1}(z)}\right) > 0, \qquad z \in \Delta. \tag{4}$$

This in conjunction with equation (3) will imply

$$\operatorname{Re}\left(\frac{zF'_{\lambda_1}(z)}{F_{\lambda_2}(z) - F_{\lambda_1}(z)}\right) > 0, \qquad z \in \Delta$$

and

$$\operatorname{Re}\left(\frac{1}{h(z,t)}\right) > 0, \qquad z \in \Delta$$

since $\operatorname{Re}\psi(z) > 0$, $z \in \Delta$. We have shown that $F_{\lambda_1} \prec F_{\lambda_2}$ on each disk $\Delta_r = \{z : |z| < r\}$ if, and only if, inequality (4) holds on that disk.

We proceed as follows. Since $F'_{\lambda_2}(0) = \lambda_2/(1+\lambda_2) > \lambda_1/(1+\lambda_1) = F'_{\lambda_1}(0)$, there exists an $r > 0$ such that $F_{\lambda_1} \prec F_{\lambda_2}$ on Δ_r. Choose r as large as possible such that $F_{\lambda_1} \prec F_{\lambda_2}$ on Δ_r, but $F_{\lambda_1}(z_1) = F_{\lambda_2}(z_2)$ for some z_1, z_2 with $|z_1| = |z_2| = r$ and assume $r < 1$.

First, suppose $z_1 = z_2$. Because $F_{\lambda_1} \prec F_{\lambda_2}$ in Δ_r, $F_{\lambda_1}(z) = F_{\lambda_2}(\omega(z))$ for some $\omega \in \mathcal{H}_0(\Delta)$ with $|\omega(z)| < |z|$ when $|z| < r$. Since $z_1 = z_2$ and $F_{\lambda_1}(z_1) = F_{\lambda_2}(z_2)$, $\omega(z_1) = z_1$. A result of Julia [2, p. 28] gives $\omega'(z_1) \geq 1$. Since $\lambda_1\lambda_2\psi(z)$ has no poles in the unit disk and since F_{λ_1} and F_{λ_2} are univalent, by equation (3), $\lambda_2 z_1 F'_{\lambda_1}(z_1) = \lambda_1 z_1 F'_{\lambda_2}(z_1) \neq 0$. Then, again using subordination in Δ_r, we have

$$\lambda_1 F'_{\lambda_1}(z_1) = \lambda_1 F'_{\lambda_2}(\omega(z_1))\omega'(z_1) = \lambda_2 F'_{\lambda_1}(z_1)\omega'(z_1).$$

However, this gives a contradiction since $\omega'(z_1) \geq 1$ but $\lambda_2 > \lambda_1$.

Second, suppose $z_1 \neq z_2$. By assumption, F_{λ_2} is convex, and therefore, the image of the circle $\partial\Delta_r$ has positive curvature everywhere. We also have that inequality (4) holds at every point on the circle $\partial\Delta_r$. Thus, the left hand side of (4) has a positive minimum ρ on $\partial\Delta_r$. Hence, on $\overline{\Delta_r}$, we have

$$\operatorname{Re}\left(\frac{zF'_{\lambda_2}(z)}{F_{\lambda_2}(z) - F_{\lambda_1}(z)}\right) > \rho > 0.$$

However,

$$\overline{\Delta_r} \subset \left\{z : \operatorname{Re}\left(\frac{zF'_{\lambda_2}(z)}{F_{\lambda_2}(z) - F_{\lambda_1}(z)}\right) > \frac{\rho}{2} > 0\right\}.$$

This implies, by the comments following inequality (4), that subordination holds in a larger circle, contradicting the choice of r. Therefore, $r = 1$, and we have the desired result. □

Additionally, we can establish a subordination relationship using the following result of Miller and Mocanu [3, p. 70] regardless of the geometric properties possessed by F_λ.

Theorem 4 (Miller, Mocanu)**.** *Let $\Phi \in K$ and let Ψ be analytic in Δ, with $\operatorname{Re}\Psi(z) > 0$, $z \in \Delta$. If F is analytic in Δ and if*

$$F(z) + \Psi(z)zF'(z) \prec \Phi(z), \qquad z \in \Delta$$

then $F \prec \Phi$.

A simple manipulation of the differential equation (2) gives

$$F_\lambda(z) + \left(\frac{1}{\lambda\psi(z)}\right) zF'_\lambda(z) = \frac{\phi(z)}{\psi(z)}. \tag{5}$$

Therefore, we have the following corollary.

Corollary 5. *If F_λ, ψ, and ϕ are as in differential equation (2), $\phi/\psi \in K$, and F_λ is analytic, then $F_\lambda \prec \phi/\psi$.*

The solutions F_λ, $\lambda > 0$, will be analytic if $\phi \in S^*$ by the Integral Existence Theorem given by Miller and Mocanu in [3, p. 50]. Also, as λ tends to infinity in equation (5), F_λ appears to tend to ϕ/ψ. Thus, if $\{F_\lambda\}_{\lambda>0}$ is a subordination chain and if F_λ, $\lambda > 0$, is convex, we would expect the limiting function ϕ/ψ to be convex. Because of this, it would not be very restrictive to require ϕ/ψ to be in K in the differential equation (2) if needed to obtain a subordination chain.

3 Convex Subordination Chains from the Bernardi Integral Operator

In this section, we exploit the structure of the differential equation (2) and a connection to a well-studied integral operator to provide families of subordinate solutions. This work supports the idea that convexity is critical in obtaining subordination chains. Providing further evidence of this, we will later produce a family of starlike solutions of differential equation (2) that is not a subordination chain.

3.1 Integral Form of the Solutions

Recall in the differential equation (2), ψ is analytic in Δ with positive real part, $\psi(0) = 1$, and $\phi \in \mathcal{H}_0(\Delta)$ with $\phi'(0) = 1$. Let $g \in \mathcal{H}_0(\Delta)$, $g(z) \neq 0$ for $z \in \Delta \setminus \{0\}$, be defined such that $zg'(z)/g(z) = \psi(z)$. Then a calculation gives

$$F_\lambda(z) = \frac{\lambda}{[g(z)]^\lambda} \int_0^z \frac{\phi(w)}{\psi(w)} [g(w)]^{\lambda-1} g'(w)\, dw. \tag{6}$$

By making the substitution $w = tz$, $0 \le t \le 1$ in (6), F_λ, $\lambda > 0$, is independent of the choice of the branch of the exponential function. In this form, we now see that $F_\lambda = \lambda/(\lambda+1) I_{g,\lambda}[\phi/\psi]$, $\lambda > 0$, where $I_{g,\lambda}$ is defined for $f \in \mathcal{H}_0(\Delta)$ by

$$I_{g,\lambda}[f](z) = \frac{\lambda+1}{[g(z)]^\lambda} \int_0^z f(w)[g(w)]^{\lambda-1} g'(w)\, dw. \tag{7}$$

A summary of several results on $I_{g,\lambda}$ which frequently rely on the Briot-Bouquet differential equation can be found in [3].

3.2 Solutions Subordinate to ϕ

Let $f \in \mathcal{H}_0(\Delta)$ where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. For $c = 1, 2, 3, \ldots$, Bernardi [1] defined

$$I_{z,c}[f](z) = (c+1)z^{-c} \int_0^z w^{c-1} f(w)\, dw = z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right) a_n z^n \tag{8}$$

which generalizes the Libera integral operator. By letting $\psi(z) = 1$ in the differential equation (2), we see $F_\lambda = I_{z,\lambda}[\phi]$, $\lambda = 1, 2, 3, \ldots$. Bernardi [1] showed $I_{z,c}[f]$ preserved some geometric characteristics of the function f. In addition, a subordination relation was established. A sequence of complex numbers, $\{b_n\}_{n=1}^{\infty}$, is a subordinating factor sequence if for $g(z) = \sum_{n=1}^{\infty} b_n z^n$ and each $f \in K$, $f * g \prec f$. The main results from [1] are summarized in the following theorem.

Theorem 6. *The function $I_{z,c}[f]$ as defined in* (8) *is convex, starlike, or close-to-convex whenever f is convex, starlike, or close-to-convex, respectively. In addition, the sequence $\{(c+1)/(c+n)\}_{n=1}^{\infty}$ is a subordinating factor sequence. In other words, if $f \in K$, $I_{z,c}[f] \prec f$.*

Additionally, Miller and Mocanu [3, p. 67] proved the Bernardi integral operator preserves the families of convex, starlike, and close-to-convex functions for c such that $\operatorname{Re} c \geq 0$. Thus, we have the following theorem.

Theorem 7. *In the differential equation (2), let $\psi(z) = 1$ and $\phi \in K$ where $\phi(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then*

$$\frac{\lambda+1}{\lambda} F_\lambda(z) = (\lambda+1)z^{-\lambda} \int_0^z w^{\lambda-1} \phi(w)\, dw = z + \sum_{n=2}^{\infty} \left(\frac{\lambda+1}{\lambda+n} \right) a_n z^n.$$

Furthermore,

$$F_{\lambda_1} \prec F_{\lambda_2} \prec \phi, \qquad 0 < \lambda_1 < \lambda_2.$$

Hence, in this case, $\{F_\lambda\}_{\lambda>0}$ is a subordination chain and $\{(\lambda+1)/(\lambda+n)\}_{n=1}^{\infty}$ is a subordinating factor sequence for $\lambda > 0$.

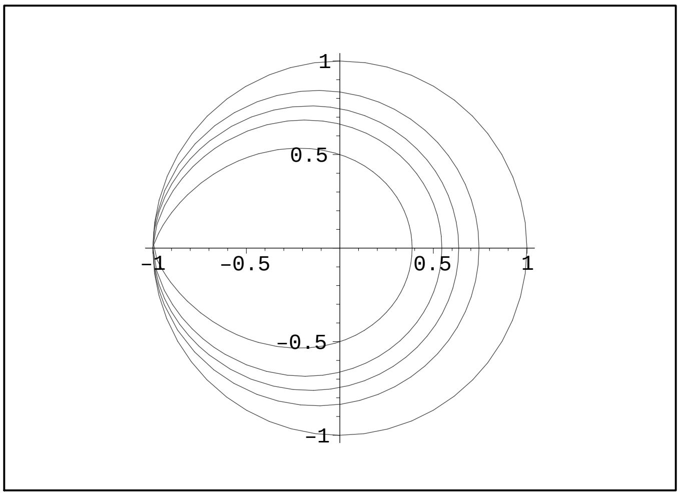

Figure 1: Let $\psi(z) = 1/(1+z)$ and $\phi(z) = z/(1+z)$. Above are the graphs of $F_\lambda(e^{i\theta})$ for $\lambda = 1$, 2, 3, 5.2 and the unit circle.

3.3 Solutions Subordinate to the Identity Function

Theorem 8. *Let $\phi(z) = z/(1+\gamma z)$ and $\psi(z) = 1/(1+\gamma z)$, $|\gamma| \le 1$, in the differential equation (2). Then F_λ, $\lambda > 0$, is convex and*

$$F_{\lambda_1} \prec F_{\lambda_2} \prec I, \qquad 0 < \lambda_1 < \lambda_2$$

where $I(z) = z$.

See Figure 1 for an example. To prove this theorem, we will use the following lemmas.

Lemma 9. *Let $\phi(z) = z/(1+\gamma z)$ and $\psi(z) = 1/(1+\gamma z)$, $|\gamma| \le 1$, in the differential equation (2), and let $a(n,\lambda)$ be the coefficient of z^n in the expansion of F_λ, $\lambda > 0$. Then*

$$\frac{\lambda}{\lambda+1} a(n,\lambda+1) = \left(\frac{\lambda+1}{n+\lambda+1}\right) a(n,\lambda).$$

Proof. If F_λ is a solution of the differential equation (2) with ϕ and ψ as given above and $|\gamma| \le 1$, $a(n,\lambda)$ must satisfy the difference equation

$$(n+\lambda+1)a(n+1,\lambda) + n\gamma a(n,\lambda) = 0.$$

Solving this gives

$$a(n,\lambda) = \frac{(-1)^{n+1}\gamma^{n-1}(n-1)!\lambda\Gamma(\lambda+1)}{\Gamma(n+\lambda+1)}.$$

Of course, this result also applies when λ is replaced by $\lambda + 1$. $\square$

Lemma 10. *Let $\phi(z) = z/(1+\gamma z)$ and $\psi(z) = 1/(1+\gamma z)$, $|\gamma| \leq 1$, in the differential equation (2). For $\lambda > 0$, if the solution F_λ is convex, then $F_{\lambda+1}$ is convex.*

Proof. By Lemma 9, we know

$$F_{\lambda+1}(z) = \frac{(\lambda+1)^2}{\lambda(\lambda+2)} \sum_{n=1}^{\infty} \frac{(\lambda+1)+1}{(\lambda+1)+n} a(n,\lambda) z^n.$$

Thus, by the comments following Theorem 6, $F_{\lambda+1}$ is convex whenever F_λ is convex. $\square$

By Lemma 10, if F_λ is convex for $0 < \lambda \leq 1$, then $\{F_\lambda\}_{\lambda>0}$ is a subordination chain by Theorem 2. In [7], Selinger proved if $g(z) = z/(1+\gamma z)$, $|\gamma| \leq 1$, and $0 < \lambda \leq 1$, $I_{g,\lambda}[K] \subset K$. Using this and the above lemmas, we now prove Theorem 8.

Proof of Theorem 8. Let $\psi(z) = 1/(1+\gamma z)$. Then $g(z) = z/(1+\gamma z)$ in equation (6). By Theorem 6 in [7], F_λ is convex for $0 < \lambda \leq 1$. By Lemma 10, F_λ is convex when $\lambda > 0$. Lastly, by Theorem 2 and Theorem 4,

$$F_{\lambda_1} \prec F_{\lambda_2} \prec I, \qquad 0 < \lambda_1 < \lambda_2.$$

$\square$

4 A Starlike Example

We conclude by presenting a case where the ratio ϕ/ψ and the solutions of differential equation (2) are starlike, but the family of solutions is not a subordination chain. Additionally, Theorem 4 fails to extend for these starlike functions. This case demonstrates the importance of convexity in obtaining subordination chains from families of solutions of differential equation (2).

Let $\psi(z) = (1-z)/(1+z)$ and let $\phi(z) = z/(1+z)^2$. Notice the similarity between differential equation (1) and differential equation (2) with this choice of ψ and ϕ. Write the solutions $F_\lambda(z) = Q_\lambda(z)/(1+z)$ where

$Q_\lambda(z) = \sum_{n=1}^\infty b_n z^n$. Since F_λ is a solution of the differential equation (2), Q_λ must satisfy

$$(1+z)zQ'_\lambda(z) + (\lambda - (\lambda+1)z)Q_\lambda(z) = \lambda z,$$

and hence, the coefficients b_n of Q_λ must satisfy the difference equation

$$(n+1+\lambda)b_{n+1} + (n-1-\lambda)b_n = 0.$$

By solving the difference equation, we have

$$Q_\lambda(z) = \sum_{n=1}^\infty \frac{\lambda\Gamma^2(\lambda+1)}{\Gamma(\lambda+1+n)\Gamma(\lambda+2-n)} z^n. \tag{9}$$

The difference equation for the coefficients of Q_λ is not too dissimilar from the difference equation that the coefficients of the de la Vallée Poussin means, V_λ, $\lambda > 0$, satisfy, and as it turns out, there is an interesting relationship between Q_λ and V_λ. Solving the difference equation derived from differential equation (1), we obtain

$$V_\lambda(z) = \sum_{n=1}^\infty \frac{\Gamma^2(\lambda+1)}{\Gamma(\lambda+1+n)\Gamma(\lambda+1-n)} z^n. \tag{10}$$

By the properties of the Gamma function, we have

$$\begin{aligned}
\frac{1+z}{z} Q_\lambda(z) &= \sum_{n=1}^\infty \frac{\lambda\Gamma^2(\lambda+1)}{\Gamma(\lambda+1+n)\Gamma(\lambda+2-n)} z^{n-1} \\
&\quad + \sum_{n=1}^\infty \frac{\lambda\Gamma^2(\lambda+1)}{\Gamma(\lambda+1+n)\Gamma(\lambda+2-n)} z^n \\
&= \frac{\lambda}{\lambda+1} + 2\lambda(\lambda+1) \sum_{n=1}^\infty \frac{\Gamma^2(\lambda+1)}{\Gamma(\lambda+2+n)\Gamma(\lambda+2-n)} z^n \\
&= \frac{\lambda}{\lambda+1} + \frac{2\lambda}{\lambda+1} \sum_{n=1}^\infty \frac{\Gamma^2(\lambda+2)}{\Gamma(\lambda+2+n)\Gamma(\lambda+2-n)} z^n \\
&= \frac{2\lambda}{\lambda+1}\left(\frac{1}{2} + V_{\lambda+1}(z)\right).
\end{aligned}$$

Thus,

$$Q_\lambda(z) = \left(\frac{2\lambda}{\lambda+1}\right)\left(\frac{z}{1+z}\right)\left(\frac{1}{2} + V_{\lambda+1}(z)\right),$$

and

$$\frac{zQ_\lambda'(z)}{Q_\lambda(z)} = \frac{1}{1+z} + \frac{zV_{\lambda+1}'(z)}{1/2 + V_{\lambda+1}(z)}.$$

Taking $f(z) = z/(1-z)$ in Theorem 1, we see $V_{\lambda+1}(\Delta)$ is starlike with respect to $-1/2$. From this, it is easily shown that Q_λ is starlike of order $1/2$. That is, $\operatorname{Re}(zQ_\lambda'(z)/Q_\lambda(z)) > 1/2$. To conclude that F_λ is starlike, we will use the following known lemma which is easily verified.

Lemma 11. *Let $F(z) = f(z)g(z)/z$. If f and g are starlike of order $1/2$, then F is starlike.*

Thus, we have the following.

Theorem 12. *Let $\psi(z) = (1-z)/(1+z)$ and $\phi(z) = z/(1+z)^2$ in the differential equation (2). Then F_λ, $\lambda > 0$, is starlike.*

Proof. Since

$$F_\lambda(z) = \frac{Q_\lambda(z)}{1+z} = \frac{\frac{z}{1+z}Q_\lambda(z)}{z},$$

and Q_λ and $z/(1+z)$ are starlike of order $1/2$, the function F_λ is starlike by Lemma 11. □

Remark 1. *From the work above,*

$$F_\lambda(z) = \frac{2\lambda}{\lambda+1}\frac{z}{(1+z)^2}\left(\frac{1}{2} + V_{\lambda+1}(z)\right).$$

Therefore, $\operatorname{Re} F_\lambda(e^{i\theta}) > 0$ because $z/(1+z)^2$ is real and positive for $z = e^{i\theta}$, $z \neq -1$, and $V_\lambda(\Delta) \subset \{z : \operatorname{Re} z > -1/2\}$ for $\lambda > 0$. Since F_λ has an infinite discontinuity at $z = -1$ and is starlike, as the argument θ increases from zero through 2π, $F_\lambda(re^{i\theta})$, $0 \leq r < 1$, must take on all values to the left of $F_\lambda(e^{i\theta})$. Thus, $F_\lambda \not\prec \phi/\psi$ where $\phi(z)/\psi(z) = z/(1-z^2)$ is the two slit mapping with slits on the imaginary axis beginning at $\pm i/2$. Not only is the solution F_λ not subordinate to the limiting function ϕ/ψ, but also, a calculation shows $F_{\lambda_1} \not\prec F_{\lambda_2}$ for $0 < \lambda_1 < \lambda_2$. See Figure 2.

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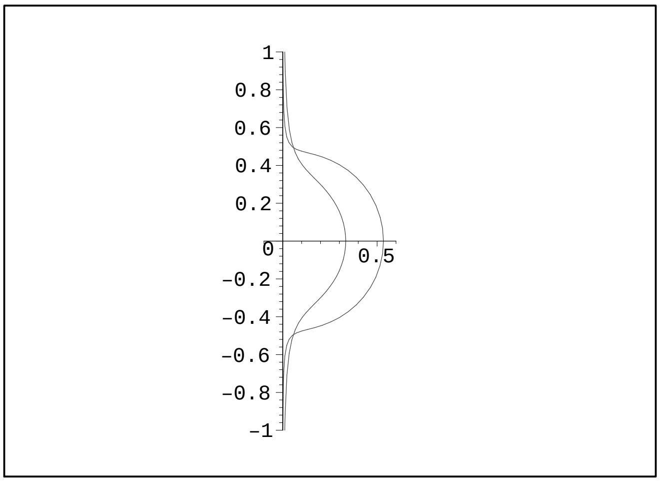

Figure 2: Let $\psi(z) = (1-z)/(1+z)$ and $\phi(z) = z/(1+z)^2$. Above are the graphs of $F_1(e^{i\theta})$ and $F_2(e^{i\theta})$. Observe $F_1 \not\prec F_2$.

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