

2. TIME VALUE OF MONEY

Objectives: After reading this chapter, you should be able to

1. Understand the concept of compounding and discounting.
2. Calculate the present value, or the future value of a single payment, or a series of payments.
3. See the effect of monthly, daily, or continuous compounding, or discounting.

2.1 [Video 02A](#), Single Payment Problems

Suppose someone offers you to have \$1000 today, or to receive it a year from now. You would certainly opt to receive the money right away. A dollar in hand today is worth more than a dollar you expect to receive a year from now. There are several reasons why this is so.

First, you can invest the money and make it grow. For example, in July 2009, the rate of interest paid by leading banks for a one-year certificate of deposit was 2%. This means you can get the \$1000 today, invest it, and earn \$20 on it by next year. There is very little risk in this investment.

Second, there is the risk of inflation that eats away the purchasing power of the dollar. The rate of inflation in USA has been rather low recently, less than 1%, but in some other countries, it has been much higher. Still you will need more than \$1 next year to buy what one dollar can buy today.

Third, there is an element of risk in this deal. If you receive the money today, you are sure to have it in your pocket. On the other hand, the person who is promising you the money may not be around next year, or he may change his mind.

Finally, you may take the money now and use it to buy some necessary things, such as food and clothing. If you already have all the necessities, you may want to spend the money on pleasurable pastimes, such as a vacation or a flat-screen television set. Human beings prefer having pleasant things as soon as possible and postpone unpleasant experiences.

The banks and thrift institutions realize this and in order to attract investors' savings they offer to pay interest on deposits. Suppose you deposit \$100 in a bank that offers 6% annual interest. This amount will become \$106 by next year, that is, it increases by a factor of 1.06. After two years it will grow by a factor of 1.06 again and it becomes $100(1.06)(1.06) = \$112.36$. The additional \$0.36 is the interest earned in the second year on the \$6 first year interest. In this way, compounding the interest annually, \$100 will grow to $100(1.06)^3 = \$119.10$, after three years.

To get a general result, let assume that the interest rate is r ; then the amount will increase by a factor of $(1 + r)$ every year. We define

PV = present value, or the initial deposit
 FV = future value of this initial deposit

After n years,

$$FV = PV(1 + r)^n \quad (2.1)$$

This is one of the basic formulas in finance. It relates four quantities: FV the future value of a sum of money, PV the present value of that money, r the rate of growth, or interest rate per period, and n the number of periods. For instance, n could be 8 years, and r could be 7% per year.

If the compounding is quarterly, then the rate of interest will be $r/4$ per quarter, but there will be $4n$ periods for compounding. Thus (2.1) becomes

$$FV = PV(1 + r/4)^{4n}$$

For monthly compounding, the rate of interest is $r/12$ and the number of periods $12n$. In general, if we carry out the compounding k times a year, then we may write (2.1) as

$$FV = PV(1 + r/k)^{kn} \quad (2.2)$$

If k becomes very large, then the procedure will compound the money "continuously." Recall the definition of the exponential function

$$e^r = \lim_{k \rightarrow \infty} \left(1 + \frac{r}{k}\right)^k \quad (2.3)$$

which gives

$$e^{rn} = \lim_{k \rightarrow \infty} \left(1 + \frac{r}{k}\right)^{kn}$$

When k approaches infinity, we see that

$$FV = PV e^{rn} \quad (2.4)$$

How is it possible to compare two cash payments that we will receive at different points in time? For example, which is more desirable: \$200 that we will get after 2 years, or \$250 available after 3 years? We find the answer, of course, by comparing their *present values*. The present value of different future payments brings them to a common level, namely, the present instant, and thus it is easy to compare them. To find the present value we rewrite (2.1) as

$$PV = \frac{FV}{(1 + r)^n} \quad (2.5)$$

The above equation represents a very useful concept in finance, namely, *discounting*. If we know the future value of a sum of money, we discount it to get its present value.

Examples

2.1. Wilkins Micawber has just received \$20,000 and he is thinking of saving it for his retirement that is 15 years away. First National Bank offers 7% interest on a 15-year deposit, compounded annually. Second National Bank gives 6.9% annually, but compounds it monthly. Third National Bank pays 6.5%, but compounds the interest continuously. What should Micawber do?

Use (2.1) to find the final value of the deposit for the first bank as

$$FV(\text{First}) = 20,000(1.07)^{15} = \$55,180.63$$

For the Second Bank, one has to use the monthly rate of interest, which is $6.9\%/12 = .069/12$. The money grows at this rate for $12 \cdot 15 = 180$ months. The future value is

$$FV(\text{Second}) = 20,000(1 + .069/12)^{180} = \$56,135.48$$

Many processes in nature show a continuous rate of growth. For example, the population of a city grows continuously, but not uniformly. If the interest is added every instant in time and added back to the principal, then the sum of money will grow continuously. In mathematics, one can model continuous growth with the exponential function, hence the term exponential growth. The exponential function y , has the form $y = e^x$, where x is the exponent and e has the approximate value 2.718281828. Use equation (2.4).

$$FV(\text{Third}) = 20,000e^{.065 \cdot 15} = \$53,023.34$$

To solve the problem in Excel, set up the following table

	A	B	C	D
1	Initial amount, \$ =	20000	Time, years =	15
2		Rate	Compounding	Final value
3	First Nat Bank	.07	Annually	=B1*(1+B3)^D1
4	Second Nat Bank	.069	Monthly	=B1*(1+B4/12)^(D1*12)
5	Third Nat Bank	.065	Continuously	=B1*exp(B5*D1)

To verify the results at [WolframAlpha](#), try these expressions.

[WRA](#) $20000 \cdot 1.07^{15}$

[WRA](#) $20000 \cdot (1 + .069/12)^{180}$

[WRA](#) $20000 \cdot \exp(.065 \cdot 15)$

He should put his money in Second National Bank, which gives the highest final value. ♥

2.2. Find the future value of the following investments:

(A) A deposit of \$10,000 left in the bank for 10 years and accumulating interest at the rate of 9% annually, compounded quarterly.

If the compounding is quarterly, then the rate of interest will be $r/4$ per quarter, but there will be $4n$ periods for compounding. Thus (2.1) becomes

$$FV = PV (1 + r/4)^{4n}$$

Or, $FV = 10,000(1 + .09/4)^{40} = \$24,351.89$ ♥

(B) A \$10,000 certificate of deposit maturing in 10 year, with 9% interest rate, compounded continuously.

For continuous compounding, use

$$FV = PV e^{rn} \quad (2.4)$$

Put $PV = \$10,000$, $r = .09$, and $n = 10$,

$$FV = 10,000 e^{(.09)(10)} = \$24,596.03$$
 ♥

To verify the results at [WolframAlpha](#), try these expressions.

[WRA](#) `10000*(1+.09/4)^40`

[WRA](#) `10000*exp(.09*10)`

2.3. You have a choice of receiving \$10,000 now, or \$11,100 a year from now, or \$12,500 two years from now. What is the best option? Assume that the discount rate is 10%.

By far the best way to compare the value of one payment and another one is to compare their *present values*. This is a very important principle in finance and we will use it frequently in the remaining chapters of the textbook. With a discount rate, $r = 10\%$, and using (2.5), the present values are:

$$\begin{aligned} \text{PV}(\text{Option I}) &= \$10,000 \\ \text{PV}(\text{Option II}) &= 11,100/1.1 = \$10,090.91 \\ \text{PV}(\text{Option III}) &= 12,500/1.1^2 = \$10,330.58 \end{aligned}$$

Comparing them, the third option is the best option. ♥

2.2 [Video 02B](#), Multiple Payment Problems

There are many examples in finance where we have to deal with a series of payments. For example, the paychecks that we receive over the course of a year, or the rent payments from a rental property, or the payments we have to make to pay off an

installment loan. In a typical problem, we have to find the present value of a set of future payments, or the final value of an account where we have made periodic deposits.

A series of payments constitute an annuity, even if the payments are not made annually. Similarly, a perpetuity is a stream of payments that goes on forever. For a series of payments, either we sum them one by one, or apply the result for the summation of a geometric series. We define a geometric series as

$$S = a + ax + ax^2 + ax^3 + \dots + ax^{n-1} \quad (1.3)$$

where a is the initial term and x is the ratio between successive terms. The summation of the series, S_n is

$$S_n = \frac{a(1-x^n)}{1-x} \quad (1.4)$$

A very important problem in finance is that of finding the present value of a set of future payments. Suppose we represent each payment, or cash flow, by C , and the discount rate by r . Then the discounted value of n such cash flows is

$$PV = \sum_{i=1}^n \frac{C}{(1+r)^i} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots + \frac{C}{(1+r)^n}$$

To find the sum of this series, we notice that the first term is $a = \frac{C}{1+r}$ and the ratio between the terms is $x = \frac{1}{1+r}$. By using equation (1.4) we get

$$PV = \frac{\frac{C}{1+r} \left(1 - \frac{1}{(1+r)^n} \right)}{1 - \frac{1}{1+r}}$$

After some simplification, it gives the result

$$\sum_{i=1}^n \frac{C}{(1+r)^i} = \frac{C[1 - (1+r)^{-n}]}{r} \quad (2.6)$$

To check equation (2.6), send the following instruction to [WolframAlpha](#),

[WRA](#) `sum(C/(1+r)^i, i=1..n)`

To find the value of an infinite geometric series, we need the relation

$$S_\infty = \frac{a}{1-x} \quad (1.5)$$

The above equation is valid only if the ratio $x < 1$, then x^n approaches zero when n approaches infinity.

For instance, we want to find the PV of a perpetuity that pays C per year forever. With a discount rate r , the result is

$$\sum_{i=1}^{\infty} \frac{C}{(1+r)^i} = \frac{C}{r} \quad (2.7)$$

In equations (2.6) and (2.7), we assume that the first cash flow occurs after one period. Now consider the case when the first cash flow is after k periods, and continues for the next n periods. The present value of the cash flows is thus

$$\frac{C}{(1+r)^k} + \frac{C}{(1+r)^{k+1}} + \frac{C}{(1+r)^{k+2}} + \dots + \frac{C}{(1+r)^{k+n-1}}$$

Write it as

$$\frac{1}{(1+r)^{k-1}} \left[\frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots + \frac{C}{(1+r)^n} \right] = \frac{1}{(1+r)^{k-1}} \sum_{i=1}^n \frac{C}{(1+r)^i}$$

Thus the present value of n cash flows each one C , the first one available after k periods is

$$PV = \frac{1}{(1+r)^{k-1}} \sum_{i=1}^n \frac{C}{(1+r)^i} \quad (2.8)$$

In many instances, we have to find the *future value* of a series of payments. An example is the future value of a retirement account in which a person makes periodic payments and the money is growing at a compound rate. Suppose, we make an initial deposit of C right now, and then another similar one after one month, and so on for a total of n deposits. Assume that the interest rate, or the rate of growth of money, is uniformly r per month. The future value of the first deposit after n months will be $C(1+r)^n$. The future value of the second deposit will be $C(1+r)^{n-1}$ because it has only $n-1$ months to grow. By extending the argument, we can find the future value of all n deposits as

$$C(1+r)^n + C(1+r)^{n-1} + C(1+r)^{n-2} + C(1+r)^{n-3} + \dots + C(1+r)$$

This is a geometric series with $a = C(1+r)^n$, $x = 1/(1+r)$, and $n = n$. Substituting these values in the general summation formula (1.4), we get

$$FV = \frac{C(1+r)^n [1 - 1/(1+r)^n]}{1 - 1/(1+r)}$$

With some simplification, it becomes

$$FV = \frac{C(1+r)[(1+r)^n - 1]}{r} \quad (2.9)$$

Equation (2.9) gives the future value of n payments made at the *beginning* of every period, each one C , which are accumulating interest at the periodic rate r . To verify (2.9), use the following instruction at [WolframAlpha](#).

`WRA sum(C*(1+r)^i, i=1..n)`

Examples

2.4. An investor deposits \$100 at the *beginning* of each month in a savings account. The bank pays 6% annual interest, but compounds it monthly. Find the total amount in this account after 100 months.

The compounding rate is 0.5% monthly, or 0.005 per month. The first \$100 are compounded for 100 months, the second \$100 for 99 months, and the last \$100 for only one month, the total amount is

$$FV = 100(1.005)^{100} + 100(1.005)^{99} + 100(1.005)^{98} + \dots + 100(1.005)$$

This is a geometric series, with first term $a = 100(1.005)^{100}$, ratio between the terms $x = 1/1.005$, and $n = 100$ terms altogether. Using equation (1.4) we get

$$FV = \frac{100 (1.005)^{100} [1 - 1.005^{-100}]}{1 - 1/1.005} = \$12,998.04 \heartsuit$$

Another way to look at the problem is to use (2.9)

$$FV = \frac{C(1+r)[(1+r)^n - 1]}{r} \quad (2.9)$$

Which gives the result as $FV = \frac{100(1.005)[1.005^{100} - 1]}{.005} = \$12,998.04 \heartsuit$

Look at the summation as

$$FV = \sum_{i=1}^{100} 100(1.005)^i$$

To verify the result, use the Maple command as

`sum(100*1.005^i, i=1..100);`

The result comes out to be \$12,998.04.

You can get the same result on [WolframAlpha](#) by sending the following instruction.

`WRA sum(100*1.005^i, i=1..100)`

Video 02C 2.5. Vinson Massif is 24 years old. He has just started a savings program. He would like to accumulate \$2 million for his retirement at the age of 65. The savings account pays interest at the annual rate of 9%, compounding it monthly. How much money should Vinson put at the beginning of each month so that at the end of 41 years he would attain his goal?

He will retire after 41 years and thus there are $12(41) = 492$ monthly payments. Suppose each payment is X . The first payment will accumulate interest for 492 months, the second for 491 months, and the last one for one month. The monthly interest rate is $0.75\% = 0.0075$. Thus

$$2,000,000 = X(1.0075)^{492} + X(1.0075)^{491} + X(1.0075)^{490} + \dots + X(1.0075)$$

This is a geometric series with $a = X(1.0075)^{492}$, $x = 1/1.0075$, and $n = 492$. Thus by equation (1.4),

$$2,000,000 = \frac{X(1.0075)^{492} [1 - 1/1.0075^{492}]}{1 - 1/1.0075}$$

This gives us $X = \$386.74$ ♥

To solve the problem in Maple, proceed as follows.

```
2000000=sum(X*1.0075^i,i=1..492);
solve(%);
```

To solve the problem on [WolframAlpha](#), copy and paste the following line:

```
WRA 2000000=sum(X*1.0075^i,i=1..492)
```

One can do the problem on Excel by the following steps. Adjust the value in the green cell B7 to get the final amount in cell B8 to be equal to the target amount in cell B4.

	A	B
1	Current age, years	24
2	Retirement age, years	65
3	Interest rate, per year	9%
4	Target amount, \$	2,000,000
5	Total months	=12*(B2-B1)
6	Monthly rate	=B3/12
7	Monthly deposit, \$	386.74
8	Final amount, \$	=B7*((1+B6)^B5-1)/(1-1/(1+B6))

2.6. Axel Heiberg has just accepted a job with an annual salary of \$24,000. He has decided to put 10% of his gross monthly income into a retirement account at the beginning of every month. The retirement account pays interest at the rate of 0.75% every month. Axel also expects to receive an annual raise of 10% each year for the next several

years. How much money will he accumulate in his retirement fund at the end of two years?

His pay is \$2000 per month and 10% of that is \$200. For the first year, the saving is \$200 per month, but second year it goes up by 10% to \$220 per month. The future value is thus

$FV = 200(1.0075)^{24} + 200(1.0075)^{23} + \dots 12 \text{ terms} + 220(1.0075)^{12} + 220(1.0075)^{11} + \dots 12 \text{ terms}$
 The above expression is a sum of two geometric series. For the first series, $a = 200(1.0075)^{24}$, $n = 12$, $x = 1/1.0075$, and for the second series, $a = 220(1.0075)^{12}$, $n = 12$, and $x = 1/1.0075$. Using (1.4), we get the answer as

$$FV = \frac{200 (1.0075)^{24} [1 - 1/1.0075^{12}]}{1 - 1/1.0075} + \frac{220 (1.0075)^{12} [1 - 1/1.0075^{12}]}{1 - 1/1.0075}$$

$$= \frac{[200 (1.0075)^{24} + 220 (1.0075)^{12}] [1 - 1/1.0075^{12}]}{1 - 1/1.0075} = \$5,529. \heartsuit$$

To solve the problem on [WolframAlpha](#), copy and paste the following line:

[WRA](#) `sum(200*1.0075^i, i=13..24)+sum(220*1.0075^i, i=1..12)`

2.7. Suppose you deposit \$200 at the beginning of every month in an account that pays interest at the rate of 9% per year, compounding it monthly. How long will it take you to accumulate \$10,000 in this account?

The monthly interest rate is 0.75% or 0.0075, and the monthly growth factor is 1.0075. Suppose it takes n months to accumulate the desired amount. Then

$$FV = 10,000 = 200(1.0075)^n + 200(1.0075)^{n-1} + \dots + 200(1.0075)$$

We may sum up the series by using equation (1.4). In our case $a = 200(1.0075)^n$, $x = 1/1.0075$, and $n = n$.

$$10,000 = \frac{200 (1.0075)^n (1 - 1/1.0075^n)}{1 - 1/1.0075}$$

Or,

$$10,000 = \frac{200 (1.0075^n - 1)}{1 - 1/1.0075}$$

Or,

$$\frac{10\,000(1 - 1/1.0075)}{200} + 1 = 1.0075^n$$

Or,

$$1.0075^n = 1.372208435$$

Taking the natural logarithm of both sides, we get

$$n \ln(1.0075) = \ln(1.372208435)$$

which gives $n = 42.347539$.

At the end of 42 months, the amount in the account is

$$\frac{200 (1.0075) (1 - 1.0075^{42})}{1 - 1.0075} = \$9904.39$$

Another \$95.61 on the first day of the 43rd month will do it. ♥

To do the problem on Excel, use (2.9) to put the accumulated amount in cell B5 and set the spreadsheet as follows. Since $10,000/200 = 50$, the answer is 50 months if the bank pays no interest. If the bank is paying interest, the answer should be somewhat less than 50 months; perhaps between 40 and 50 months. Start by putting 40 in cell B3.

	A	B	C
1	Monthly payment, C	200	
2	Monthly interest rate, r	=.09/12	
3	Number of months, n	42	
4	Target amount	10000	
5	Accumulated amount	=B1*(1+B2)*((1+B2)^B3-1)/B2	
6	Shortage	=B4-B5	last month's payment
7	Answer:	=B3+1	months

Adjust the value in cell B4 until the shortage in cell B6 is less than 200. The answer is 43 months, with the last monthly payment being \$95.61.

To solve the problem on [WolframAlpha](#), copy and paste the following:

[WRA](#) `10000=sum[200*1.0075^i,i=1..n]`

2.8. Suppose you deposit \$125 at the beginning of each month in an account that compounds interest *continuously* at the annual rate of 8%. Find the total amount in this account after 30 months.

By using equation (2.4), the *FV* of the first \$125 after 30 months will be $125e^{.08(30/12)}$. Similarly, the *FV* of the second \$125 after 29 months will be $125e^{.08(29/12)}$, and so on. The total amount will be

$$FV = 125e^{.08(30/12)} + 125e^{.08(29/12)} + 125e^{.08(28/12)} \dots 30 \text{ terms}$$

This series can be summed by using (1.4), where $n = 30$, $a = 125e^{.08(30/12)} = 152.6753448$, $x = e^{-.08/12} = .9933555063$ and $n = 30$. Thus

$$FV = \frac{125e^{.08(30/12)} (1 - e^{-.08(30/12)})}{1 - e^{-.08/12}} = \frac{125(e^{.08(30/12)} - 1)}{1 - e^{-.08/12}} = 4165.15$$

The account should have \$4,165.15 at the end of 30 months. ♥

To solve the problem on Excel, set up a table like this one.

	A	B	C
1	Amount of deposit =	125	dollars
2	Number of periods =	30	months
3	Continuously compounded interest rate =	=.08/12	per month
4	Final amount =	=B1*(exp(B3*B2)-1)/(1-exp(-B3))	

To verify the answer at [WolframAlpha](https://www.wolframalpha.com), use the following instruction.

[WRA](#) `sum(125*exp(.08*i/12), i=1..30)`

If you want to develop a general formula with continuous compounding, you may do it as follows. Suppose you deposit A dollars per period, at the beginning of every period, for n periods. The continuously compound rate of interest is r . Then the first deposit becomes Ae^m after n periods. The second deposit will become $Ae^{r(n-1)}$ after $n-1$ periods, the third one $Ae^{r(n-2)}$ after $n-2$ periods, and so on. The total amount will be

$$S = Ae^m + Ae^{r(n-1)} + Ae^{r(n-2)} + n \text{ terms}$$

This is a geometric series with first term $a = Ae^m$, the ratio between the terms $x = e^{-r}$, and the number of terms $n = n$. Use (1.4) to find the sum as

$$S_n = \frac{Ae^m(1 - e^{-rn})}{1 - e^{-r}} = \frac{A(e^m - 1)}{1 - e^{-r}}$$

2.9. Clifford Montdale is going to put \$100 at the beginning of each month in a savings account that pays interest at the rate of 7% per year, compounding it monthly. This rate is fixed for one year, and it will go down to 6% in the second year. Find the balance in this account after two years.

Consider the future value of each deposit, and then sum them up. The first deposit will remain in the bank earning interest at the rate of 7/12% per month for the first 12 months, and then at a rate of 6/12 = 1/2% per month for the next 12 months. Its future value is $100(1 + .07/12)^{12}(1 + .06/12)^{12}$.

The next deposit's future value will be $100(1 + .07/12)^{11}(1 + .005)^{12}$, because it will earn interest at the rate of 7/12% for the first 11 months, then at the rate of 6/12% = .5% = .005 for the next 12 months. Continuing the procedure further, the future value of the first 12 payments will be

$$100(1 + .07/12)^{12}(1.005)^{12} + 100(1 + .07/12)^{11}(1.005)^{12} + \dots + 100(1 + .07/12)^1(1.005)^{12}$$

During the second year, the interest rate is down to .5% per month. We can write the sum of future values of the payments for second year as

$$100(1.005)^{12} + 100(1.005)^{11} + 100(1.005)^{10} + \dots + 100(1.005)^1$$

This results in two geometric series. For the first series, the initial term $a = 100(1 + .07/12)^{24}(1.005)^{12}$, The ratio between the terms $x = 1/(1 + .07/12)$, and the number of terms $n = 12$. For the second series, $a = 100(1.005)^{12}$, $x = 1/(1.005)$, and the number of terms $n = 12$. Applying (1.4), we get the total amount to be

$$S = \frac{100(1 + .07/12)^{12}(1.005)^{12}[1 - 1/(1 + .07/12)^{12}]}{1 - 1/(1 + 0.07/12)} + \frac{100(1.005)^{12}[1 - 1/1.005^{12}]}{1 - 1/1.005}$$

$$= \$2563.09 \heartsuit$$

The answer is quite reasonable because it consists of \$2400 of actual deposits and \$163 in interest for two years. To solve the problem on [WolframAlpha](#), copy and paste the following line:

```
sum(100*1.005^12*(1+.07/12)^i,i=1..12)+sum(100*1.005^i,i=1..12)
```

Video 02.10 2.10. Harold Brown is planning to put some money on the first of every month in a savings account that pays 12% annual interest, compounded monthly. He will start by putting \$500 on February 1, 2008, but keep on increasing his deposits by 1% every month. On what date will this account have more than \$1 million for the first time?

This is a problem of future value and compounding. We find the final value as the sum of all the deposits with proper compounding. Suppose Harold Brown reaches the million-dollar mark after n months. The first \$500 will earn interest at the rate of 1% per month, compounding monthly, and after n months, its value will become $500(1.01)^n$.

Next month, the deposit will increase by 1% and it will be $500(1.01)$. This deposit will grow for only $n - 1$ months, because one month has already elapsed. Its final value will be $500(1.01)(1.01)^{n-1}$. We can write it as $500(1.01)^n$. The final value of both deposits will be identical because the second deposit starts out with a bigger amount, but has less time to grow. The two factors cancel out precisely.

The deposit for the following month is 1% greater than the previous month's deposit and it equals $500(1.01)^2$. However, it can grow only for $n - 2$ months and it finally becomes $500(1.01)^2(1.01)^{n-2}$. Merging the powers, the result is $500(1.01)^n$. This gives exactly the same final value. We discover that the *final* value of each deposit is the same, namely, $500(1.01)^n$. Since there are n such deposits, their total final value should be $n[500(1.01)^n]$.

We can summarize the previous discussion in the following equation.

$$1,000,000 = 500(1.01)^n + 500(1.01)(1.01)^{n-1} + 500(1.01)^2(1.01)^{n-2} + \dots$$

Or,
$$1,000,000 = 500n (1.01)^n$$

We can solve the above equation for n by using Maple. We enter the instruction

`fsolve(1000000=500*n*1.01^n,n);`

and we get $n = 221.2568054$. Thus after 222 months, or, 18 years and 6 months, he should have more than a million dollars. The exact amount is $500 \cdot 222 \cdot 1.01^{222} = \$1,010,806.30$. ♥

To get the answer on [WolframAlpha](#), enter the following

[WRA](#) `1000000 = 500*n*1.01^n`

If Harold Brown decides to increase his monthly deposit by 2% every month, while the bank keeps on paying 1% monthly interest, then the first equation will become

$$1,000,000 = 500(1.01)^n + 500(1.02)(1.01)^{n-1} + 500(1.02)^2(1.01)^{n-2} + \dots$$

In this case, the numbers do not cancel out neatly. Notice that the sum of the powers is n . The factor (1.01) starts with power n and ends with power 1. The factor (1.02) starts with power 0 and it ends with power $n - 1$. We can represent the above equation as follows

$$1,000,000 = \sum_{i=1}^n 500(1.02)^{i-1}(1.01)^{n+1-i}$$

Note that the sum of the powers is $i - 1 + n + 1 - i = n$ as required. The starting values for the two powers are 0 and n , and the first term in the summation is $500(1.02)^0(1.01)^n$, when $i = 1$. To solve the equation using Maple, enter

`1000000=sum(500*1.02^(i-1)*1.01^(n+1-i),i=1..n);
solve(%);`

The result is 162.1914189. This means Harold Brown can achieve his goal sooner, in 163 months. This is quite reasonable because he is increasing his initial deposit at a faster rate.

For [WolframAlpha](#), enter the following

`1000000=sum(500*1.02^(i-1)*1.01^(n+1-i),i=1..n)`

It does not give an accurate answer. However, the two diagrams suggest a value of around 162.

To do the problem on Excel, set up a table as follows.

	A	B
1	500	=A1*1.01
2	=A1*1.02	=(A2+B1)*1.01
3	=A2*1.02	=(A3+B2)*1.01

Column A will show the deposit for each month and column B the total accumulated in the account up to that point. Next, highlight the cells (A2,B2,A3,B3) and drag down the handle. When you reach row 163, the total accumulated in cell B163 will be 1,018,185.809, which is just over a million dollars.

2.11. The winner of the recent lottery received notice that he would get his money in 20 annual installments of \$281,347 each. He will get the first installment right now. If the discount rate is 12%, find the present value of his winnings.

$$\begin{aligned}
 PV &= 281,347 + \sum_{i=1}^{19} \frac{281,347}{1.12^i} \\
 &= 281,347 + \frac{281347(1 - 1.12^{-19})}{0.12} = \$2,353,686 \heartsuit
 \end{aligned}$$

To get the answer on [WolframAlpha](#), enter the following,

[WRA](#) `sum(281347/1.12^i, i=0..19)`

2.12. A person has the following options in settlement of a life insurance policy:

- (1) A cash settlement of \$100,000, or
- (2) A monthly payment of \$1,000 available at the end of each month for the next 20 years.

If the proper discount rate in this case is 12%, which method is better for the recipient?

The present value of the monthly payments is the sum of discounted cash flows. For monthly cash flows, the discount rate is 1% per month. There are 240 payments. For summation of terms, use (2.6).

$$\sum_{i=1}^{240} \frac{1000}{1.01^i} = \frac{1000(1 - 1.01^{-240})}{.01} = \$90,819.42$$

Clearly, the cash settlement is better. \heartsuit

2.13. You have just bought a car from your friend. He gives you two options: pay for the car in 48 monthly installments of \$109 each, or 60 monthly installments of \$89 each. The time value of money is 12% annually. Which is the cheaper method of payment?

Compare the present value of the two payment options by discounting them at the rate of 12% per year, or 1% per month.

$$PV \text{ of 48 monthly payments} = \sum_{i=1}^{48} \frac{109}{1.01^i} = \frac{109(1 - 1.01^{-48})}{.01} = \$4,139.16$$

$$PV \text{ of 60 monthly payments} = \sum_{i=1}^{60} \frac{89}{1.01^i} = \frac{89(1 - 1.01^{-60})}{.01} = \$4,001.00$$

It is better to take the 60-month option. ♥

2.14. You are interested in buying a new sports car costing \$21,000. You can afford only \$5,000 as the down payment and the bank will finance the rest at 11.8%. What are the level monthly payments that will pay off the loan in 48 months?

The basic financial principle in this problem is as follows:

$$\boxed{\text{The present value of the loan}} = \boxed{\text{The present value of future payments}}$$

We can express it as

$$L = \sum_{i=1}^n \frac{P}{(1+r)^i} \quad (2.10)$$

The amount of loan is \$16,000, the number of payments is 48, and the monthly interest rate is 0.118/12. Thus if P is the monthly payment, then

$$\sum_{i=1}^{48} \frac{P}{(1 + .118/12)^i} = \frac{P[1 - (1 + .118/12)^{-48}]}{.118/12} = 16,000$$

Or,
$$P = \frac{16,000 (0.118/12)}{1 - (1 + 0.118/12)^{-48}} = \$419.77 \text{ ♥}$$

To get the answer on [WolframAlpha](#), enter the following expression

[WRA](#) `sum(P/(1+.118/12)^i, i=1..48)=16000`

2.15. You have bought a piece of land in Wayne County for \$12,000. You agreed to pay the owner the price in five equal annual installments of \$3,000 each, the first one right now. What is the implicit rate of interest that the owner is charging?

Equating the cash price of the land to the discounted value of all the payments, we get

$$12,000 = 3000 + \sum_{i=1}^4 \frac{3000}{(1+r)^i}$$

Or,
$$9,000 = \sum_{i=1}^4 \frac{3000}{(1+r)^i}$$

To solve this equation using Maple, type the main instruction as

```
fsolve (9000=sum (3000/ (1+r) ^i, i=1..4) , r, 0..1) ;
```

The instruction "**fsolve**" requires the program to solve the following equation in floating point and find the value of r within the range 0 to 1. The instruction "**sum**" takes the summation of the following terms. After some computation the result .1258983250 shows up on the screen. This is the interest rate of 12.59%. ♥

	A	B	C
1	Number of periods =	4	
2	Each payment =	-3000	dollars
3	Present value of loan =	9000	dollars
4	Implied interest rate =	=RATE(B1,B2,B3)	13%

One can do the problem in Excel by using the RATE function. The inputs for the function are =RATE(nper,pmt,pv,[fv],[type],[guess]), but the result is inaccurate.

At [WolframAlpha](#), enter the following to get a *positive real solution*

[WRA](#) 12000=3000+sum(3000/(1+r)^i, i=1..4)

2.16. You have borrowed \$56,000 as a mortgage loan to buy a house. The bank will charge interest at the rate of 9% annually and requires a minimum monthly payment of \$500. At the end of five years, you must pay off the entire mortgage by a “balloon payment.” You plan to pay only the minimum amount each month and then pay off the loan with the final payment. Find this balloon payment.

During uncertain economic times, when the interest rates are liable to fluctuate widely, the lending institutions give out short-term loans that require a balloon payment to pay off the loan. At the time of the balloon payment, the borrower can renegotiate the loan at the current interest rates and it may include another balloon payment. Suppose the balloon payment is B , then the equality of loan value to the present value of all payments implies that

$$L = \sum_{i=1}^n \frac{P}{(1+r)^i} + \frac{B}{(1+r)^n} \quad (2.11)$$

Put numerical values,

$$56,000 = \sum_{i=1}^{60} \frac{500}{1.0075^i} + \frac{B}{1.0075^{60}}$$

Or,
$$56,000 = \frac{500(1 - 1.0075^{-60})}{.0075} + \frac{B}{1.0075^{60}}$$

Which gives $B = \left[56,000 - \frac{500(1 - 1.0075^{-60})}{.0075} \right] (1.0075^{60}) = \$49,966.07$ ♥

You can check the answer with Maple by using the following steps.


```
L=sum(P/(1+r)^i,i=1..n)+B/(1+r)^n;
subs(L=56000,P=500,r=.09/12,n=5*12,%);
solve(%);
```

The following Excel table will solve the problem.

	A	B	C
1	Interest rate =	.0075	per month
2	Number of payments =	60	
3	Payment per month =	500	dollars
4	Initial loan =	-56000	dollars
5	Balloon payment =	=FV(B1,B2,B3,B4)	\$49,966.07

To get an *approximate* answer on [WolframAlpha](#), enter the following,

[WRA](#) `56000=sum(500/1.0075^i,i=1..60)+B/1.0075^60`

2.17. You have borrowed \$500 from a friend with the understanding that you will pay him back in three installments: \$100 after one month, \$200 after two months, and \$220 at the end of the third month. Find the implied interest rate in this arrangement.

Set the loan value to the *PV* of the three payments as

$$500 = \frac{100}{1+r} + \frac{200}{(1+r)^2} + \frac{220}{(1+r)^3}$$

You cannot solve this problem by the Present Value tables but you can go to [WolframAlpha](#) and paste the following instruction.

[WRA](#) `500=100/(1+r)+200/(1+r)^2+220/(1+r)^3`

the answer comes out to be $.017777 = 1.7777\%$ monthly.

The *nominal* annual rate is 12 times the monthly rate, that is, $12(.017777) = .2133 = 21.33\%$. The *effective* annual rate is due to the monthly compounding and it comes out to be $1.017777^{12} - 1 = 23.55\%$ annually. The effective interest rate is higher than the nominal rate. ♥

2.18. Sebastian Cabot has received \$135,000 in compensatory damages. He has placed the money in a trust account that pays 6% annual interest, compounded monthly. Cabot will withdraw \$1000 a month out of this account, starting one month after the initial deposit. Calculate the money in the account after 25 withdrawals.

The monthly interest rate is $.5\% = .005$. With the withdrawal rate of \$1000 a month, the present value of 25 withdrawals is $\sum_{i=1}^{25} \frac{1000}{1.005^i}$. This means the *present value* of amount of

money in the account after 25 months will be $\left(135,000 - \sum_{i=1}^{25} \frac{1000}{1.005^i}\right)$. This amount is growing at the rate of .005 per month for 25 months. Thus, its *future value* is

$$FV = \left(135,000 - \sum_{i=1}^{25} \frac{1000}{1.005^i}\right)(1.005^{25}) = \$126,368.29 \heartsuit$$

In general, it becomes

$$FV = \left(A - \sum_{i=1}^n \frac{w}{(1+r)^i}\right)(1+r)^n \quad (2.12)$$

Here FV is the future value of the account, which had an initial amount A . The account pays interest at rate r . The owner of the account has made n withdrawals, each one equal to w . This leads us to the question that if a person has a nest egg A from which he regularly withdraws w per month, how long will it take him to exhaust his savings. To answer that, we use Maple and type in the instructions:

```
0=(A-sum(w/(1+r)^i,i=1..n))*(1+r)^n;
solve(%,n);
```

After some simplification, we get the answer as

$$n = \frac{\ln\left(\frac{w}{w - Ar}\right)}{\ln(1+r)} \quad (2.13)$$

Consider a person with a total savings of \$400,000, which is earning interest at the rate of ½% per month. He withdraws \$3000 from it every month. He will exhaust his savings in

$$n = \frac{\ln\left(\frac{3000}{3000 - 400,000 \cdot .005}\right)}{\ln(1.005)} = 220.3 \text{ months} = 18.36 \text{ years} \heartsuit$$

2.19. Alabama Corporation has taken a loan of \$60,000 with the understanding that it will make the monthly payments of \$600. The bank will charge the interest at the rate of 9% per year on the unpaid balance. After how many months will the balance become \$48,686.38?

The present value of the loan L in terms of the discounted future values is

$$L = \sum_{i=1}^n \frac{P}{(1+r)^i} + \frac{B}{(1+r)^n} \quad (2.11)$$

where P is the regular monthly payment and B is the balloon payment, or balance after n periods. We may solve it for n by using Maple as follows:

```
L=sum(P/(1+r)^i,i=1..n)+B/(1+r)^n;
solve(%,n);
```

The result is

$$n = \frac{\ln\left(\frac{rB - P}{rL - P}\right)}{\ln(1 + r)} \quad (2.14)$$

Substituting $r = 0.0075$, $B = \$48,686.38$, $L = \$60,000$, $P = \$600$, we get the value of n as

$$n = \frac{\ln\left(\frac{.0075*48,686.38 - 600}{.0075*60,000 - 600}\right)}{\ln(1.0075)} = 60$$

Thus after 60 months the balance will be \$48,686.38. ♥

To do the problem on Excel, you can set up a table like this one. Adjust the number of payments in cell **B2**, until the value in cell B5 becomes equal to required balance of \$48,686.38.

	A	B	C
1	Interest rate =	.0075	per month
2	Number of payments =	60	
3	Payment per month =	600	dollars
4	Initial loan =	-60000	dollars
5	Balance =	=FV(B1,B2,B3,B4)	\$48,686.38

To check the answer at [WolframAlpha](#), copy and paste the following instruction.

[WRA](#) `60000=sum(600/1.0075^i,i=1..n)+48686.38/1.0075^n`

2.20. A bank offers the following program to its customers. If you deposit \$100 at the beginning of every month for the next 7 years, then in return the bank will give you \$100 a month forever, starting a month after your last monthly payment. If the time value of money is 13.2% annually, would you join in this program?

First, find the present value of *your* payments. Suppose you make your first payment today, the second payment at the end of the first month, and the 84th payment at the end of the 83rd month. The monthly discount rate is $13.2/12 = 1.1\% = .011$. Because you make the first payment now, the present value of 84 payments is

$$PV_1 = 100 + \sum_{i=1}^{83} \frac{100}{1.011^i} = 100 + \frac{100(1 - 1.011^{-83})}{.011} = \$5524.33$$

Next, find the present value of *bank's* payments. You made your last payment at the end of the 83rd month and the bank will make its first payment at the end of the 84th month. The present value of the bank's payments is

$$PV_2 = \frac{100}{1.011^{84}} + \frac{100}{1.011^{85}} + \frac{100}{1.011^{86}} + \dots \infty$$

This is an infinite geometric series with $a = 100/1.011^{84}$ and $x = 1/1.011$. Using (1.5),

$$PV_2 = \frac{100/1.011^{84}}{1 - 1/1.011} = \$3666.58$$

You are paying the bank an excess amount of $5,524.33 - 3,666.58 = \$1,857.75$, calculated in present value dollars. The *net present value* of this transaction is $-\$1,857.75$. Because the net present value is negative, you should not join. ♥

You can find the net present value at [WolframAlpha](#) with the following instruction.

[WRA](#) `-100-sum(100/1.011^i,i=1..83)+sum(100/1.011^i,i=84,infinity)`

2.21. In the last problem, at what minimum rate of interest would the customers consider making their deposits in this program?

The bank's payments have a smaller present value due to a higher discount rate. One has to find a smaller discount rate that will equate these two values. At some equilibrium point, the equation $PV_1 = PV_2$ will hold. This gives us

$$1 + \sum_{i=1}^{83} \frac{1}{(1+r)^i} = \sum_{i=84}^{\infty} \frac{1}{(1+r)^i} = \frac{1}{(1+r)^{83}} \sum_{i=1}^{\infty} \frac{1}{(1+r)^i}$$

Or,
$$1 + \frac{1 - (1+r)^{-83}}{r} = \frac{1}{(1+r)^{83}} \times \frac{1}{r}$$

Multiplying by r throughout, we get

$$r + 1 - (1+r)^{-83} = \frac{1}{(1+r)^{83}}$$

Multiply by $(1+r)^{83}$ to get

$$(1+r)^{84} = 2$$

Or,
$$r = 2^{1/84} - 1 = .008285892 \text{ monthly}$$

The annual rate is $12(.008285892) = 0.099430704 = 9.943\%$ annually. If the interest rates in the market are less than 9.943%, then the depositors will come out ahead. In the time of high inflation, when the interest rate is more than 10%, the bank will win this game. ♥

To perform the calculation on [WolframAlpha](#), copy and paste the following:

$$\boxed{\text{WRA}} \quad \text{sum}(1/(1+r)^i, i=0..83) = \text{sum}(1/(1+r)^i, i=84, \text{infinity})$$

The result is .00828589.

Problems

2.22. You have decided to deposit \$100 in O'Neill National Bank at the beginning of every month. The bank compounds interest on a monthly basis but at a variable rate adjusting it annually. You know that during the first year, the interest will be 8% annually, but during the second year, it may go up to 9%. Find the expected amount in your account after two years. \$2631.00 ♥

2.23. Conrad Aiken has a retirement account in which he has been adding a certain sum of money on the first of every month. He started by depositing \$250 in the first month but kept on increasing his deposits by 1% every month. The bank was adding interest to his account monthly at the rate of 0.8% per month during the first year and then 0.9% per month during the second year. Find the total amount in this account at the end of two years. \$7,500.87 ♥

2.24. Homer Zeno has borrowed \$50,000 from the bank with the understanding that he will make a minimum payment of \$500 per month. The bank will charge interest at the rate of 1% per month on the unpaid balance. Homer plans to make \$500 monthly payments for the first 12 months, then \$600 monthly payments for the next 48 months, and then pay off the entire loan by making a balloon payment at the end of the 61st month. Find this balloon payment. \$44,316.52 ♥

2.25. You have taken a \$100,000 mortgage loan at Albert Savings Association at the rate of 9% annually. You are required to pay \$1,000 monthly payment. Approximately, how long will it take you to pay off the loan? 15 years 6 months ♥

2.26. Cooper Corporation has borrowed \$120,000 from the bank at 8% annual interest rate, compounded monthly. The company plans to pay \$2,000 per month for the first 12 months, and then pay \$2500 per month for the next 12 months. Find the remaining balance of the loan after 24 months. \$82,655.21 ♥

2.27. Atbara Corporation has taken a five-year loan from the local bank at the annual interest rate of 9%. Atbara will pay back the \$250,000 loan in monthly installments. Find (a) the monthly payment, and (b) the balance of the loan after 24 payments. (a) \$5189.59, (b) \$ 163195.99 ♥

2.28. Akron Corporation has borrowed \$1.2 million from a bank with the understanding that it will pay \$50,000 a month, until the loan is paid off. The bank will charge 9% per year interest on the unpaid balance, calculated monthly. Akron will make the payments at the end of each month. Find the following:

(A) How long will it take Akron to pay off the loan? 27 months ♥

(B) What is the balance of the loan after 24 months? \$126,272.71 ♥

2.29. You would like to accumulate a million dollars for your retirement. You have another 35 years before you retire. The local bank, where you intend to keep the money, will compound interest monthly at the annual rate of 6%. How much money should you deposit at the beginning of each month to reach your goal? \$698.41 ♥

2.30. Rutherford B. Hayes has borrowed \$80,000 as a mortgage loan at 7.5% interest rate and 30-year term. He has to pay the loan in monthly installments. After how many payments will the unpaid balance become \$40,000? 265 months ♥

Multiple Choice Questions

1. The future value of \$1100, compounding at the rate of 6% annually, after 10 years is

- (a) \$1600.00 (b) \$1790.85
(c) \$1819.40 (d) \$1969.93

2. The present value of \$5000 that you will get after 10 years, discounting at the rate of 5% per year, is

- (a) \$2508.91 (b) \$2899.77
(c) \$2965.34 (d) \$3069.57

3. Suppose Republic of Scandia has a steady 30% inflation rate and a loaf of bread costs 100 liras today. Its price, in liras, last year was

- (a) 66.67 (b) 70
(c) 76.92 (d) 130

4. The monthly interest rate on a savings account is 1%, compounded monthly. The effective annual rate is

- (a) 11.25% (b) 12.00%
(c) 12.68% (d) 13.13%

5. If the discount rate is 7%, then the present value of \$40,000 that you expect to get after 15 years is

- (a) \$14,497.84 (b) \$15,037.48
(c) \$106,400.80 (d) \$110,361.26

6. The future value of \$10,000 after 11 years, growing at the rate of 12% per year is

- (a) \$34,237.40 (b) \$34,522.71
(c) \$34,785.50 (d) \$34,984.51

7. The future value of \$1000 after 5 years, with interest rate 6% and using monthly compounding, will be

- A. \$1348.85
B. \$1338.23
C. \$1374.49
D. \$1349.86

8. The present value of \$10,000, available after 6 years, with continuous discounting at the rate 7% per year, is

- A. \$9958.09
B. \$6663.42
C. \$6570.47
D. \$6650.57

Key Terms

annuity, 21
compounding, 17, 18, 20, 23,
24, 25, 27, 28, 33
discount rate, 20, 21, 22, 30,
36
discounting, 17, 18, 31

effective annual rate, 33
exponential function, 18
future value, 17, 18, 20, 22,
23, 25, 27, 28, 34
inflation, 17, 37
nominal annual rate, 33

perpetuity, 21, 22
present value, 17, 18, 20, 21,
22, 30, 31, 32, 34, 35, 36
risk, 17