

# Homeowner's Dilemma: To Default or not to Default?

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## Abstract

This paper examines the homeowner's decision to default on a home mortgage. By doing so, he will lose his equity in the house, which can be quite substantial. Should he default as soon as the market value of the house falls below the balance on the mortgage loan? Should he wait with the expectation that the value of the house will rise rapidly in the future? The paper develops an optimal strategy in discrete time with the assumption that the home values are stochastic in nature.

## 1. Introduction

When a person wants to buy a house, he may apply for a home mortgage to finance the purchase. He needs a down payment to satisfy the bank's requirement. After the approval of the loan, he buys the house. Initially, the down payment is the difference between the market price of the house and the amount of mortgage loan.

$$D = H_0 - B_0 \quad (1.1)$$

where  $D$  is the down payment,  $H_0$  is the value of the house at time zero, and  $B_0$  is the initial amount of loan. The homeowner is holding a portfolio of two items, the house and the loan, with initial value of the portfolio  $\Pi$  being equal to  $D$ .

$$\Pi = H_0 - B_0 \quad (1.2)$$

As the balance on the loan goes down and the value of the house appreciates, the value of the portfolio increases with time. Suppose the time to maturity of the loan is  $N$  (months) and interest rate charged by the bank is  $m$  (per month). In discrete time framework, the balance of the loan at time  $n$  (months) satisfies the equation

$$B_n = \frac{B_0[(1+m)^N - (1+m)^n]}{(1+m)^N - 1} \quad (1.3)$$

Suppose the value of the house appreciates uniformly with time at a steady rate  $a$  percent per month, then after time  $n$  months, the value will become  $H_0(1+a)^n$ . We may write the value of the portfolio after time  $n$ ,  $\Pi_n$  as

$$\Pi_n = H_0(1+a)^n - \frac{B_0[(1+m)^N - (1+m)^n]}{(1+m)^N - 1} \quad (1.4)$$

Suppose a person buys a \$100,000 house with a 20% down payment. He takes a 30-year (360 months) mortgage at annual 6% interest ( $\frac{1}{2}\%$  per month). The face amount of the mortgage is \$80,000 with a monthly payment of \$479.64. Suppose the value of the house appreciates at 5% annual rate ( $\frac{5}{12}\%$  per month), increasing the value of the portfolio with the passage of time.

The home prices can also fall. If the *decline* in the prices is 5% per year, the value of the portfolio will decrease over time. It can even become negative. The value of the portfolio in both cases is shown in Figure 1 as black and red curves.

The critical factor in this case is  $a$ , the rate of change in the value of the house. With a  $\frac{5}{12}\%$  per month decline in the value of the house, the portfolio becomes  $-\$36.59$  after 77 months. A more detailed analysis shows that the value of the portfolio reaches a minimum of  $-\$9677$  at 181 months. It starts to *increase* again, reaching a value of  $\$86.71$  at 283 months, finally reaching  $\$22,243$  at 30 years. Figure 2 shows this graphically.

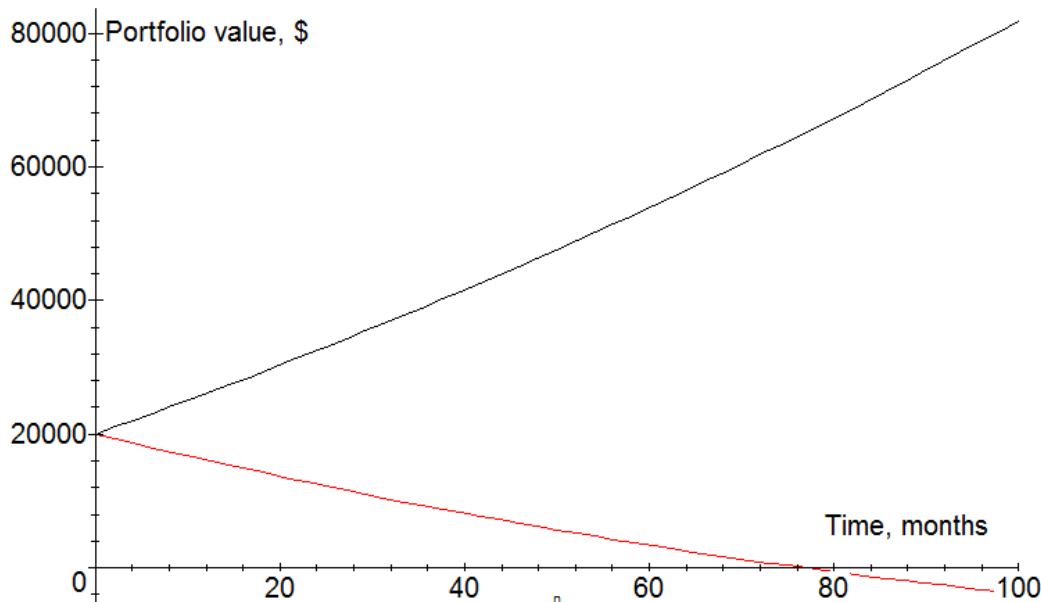


Figure 1: Portfolio value in equation (1.4) with home appreciation  $a = 5\%$  per year, black curve, and  $a = -5\%$  per year, red curve. For  $a = -5\%$  per year, the portfolio value becomes  $-\$36.59$  after 77 months.

To simplify the analysis, make the following assumptions.

1. The homeowner pays a constant amount of money every month to amortize the loan. The monthly payment is

$$P = \frac{B_0 m (1 + m)^N}{(1 + m)^N - 1}$$

2. The home value appreciates at a constant rate  $a$ ; meaning that if the house value is initially  $H_0$ , after  $n$  months, it becomes  $H_n = H_0 (1 + a)^n$ .

3. The homeowner does not have the option to prepay the loan, or to default on the loan.

4. Initially, the payment on the loan is equal to the benefit that the homeowner gets by living in the house.
5. There are no catastrophic events, such as a sudden drop in the value of the house due to a disaster, or the loss of income of the homeowner.

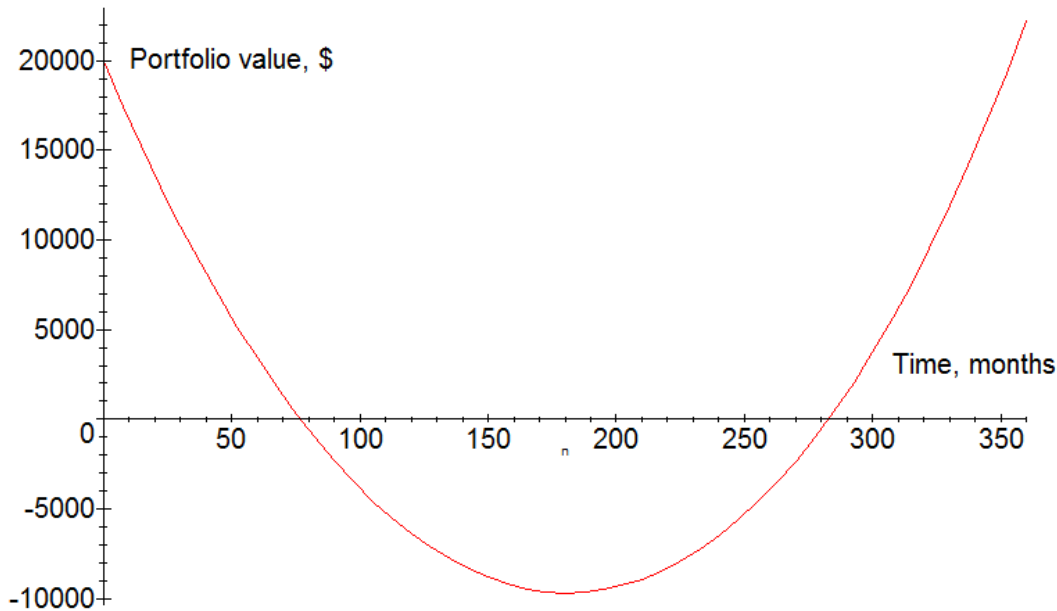


Figure 2: Portfolio value in equation (1.4) with home appreciation  $a = -5\%$  per year, red curve. The value of the portfolio reaches a minimum of  $-\$9677$  at 181 months. It starts to increase again, reaching a value of  $\$86.71$  at 283 months, finally reaching  $\$22,243$  at 30 years

## 2. Stochastic Model

To make the model more realistic, we change two of the assumptions in the previous section. First, we assume that the value of the house changes stochastically; and second, the homeowner has the option to prepay the mortgage loan, or to default on it.

Consider the case when the value of the house varies stochastically, following a Brownian motion. The implication is that the value of the house does not change uniformly; rather it fluctuates up or down every month randomly. Further, consider the possibility that the value of the house has been drifting downward and it has reached a point where the homeowner is wondering whether it is worthwhile to give the keys of the house to the bank and wipe out the mortgage loan.

At this time, we will not consider the possibility of prepayment of the loan. If the homeowner has the ability to default on the loan, he possesses a put option on the house with an exercise price equal to the current balance of the loan. One cannot use Black-Scholes model (1) to evaluate the put because of two reasons. First, the put is an American put. Second, the exercise price of the put is not constant, because it decreases with time.

Considering these factors, we may write (1.4) as

$$\Pi_n = H_n - \frac{B_0[(1+m)^N - (1+m)^n]}{(1+m)^N - 1} + P_n \quad (2.2)$$

In (2.2), the exercise price  $X$  of the put option declines with time, and after  $n$  months, it is

$$X_n = \frac{B_0[(1+m)^N - (1+m)^n]}{(1+m)^N - 1} \quad (2.3)$$

The quantity  $P_n$  is the value of an American put option, after  $n$  months, held by the homeowner. Its value depends on the value of the house, the balance of the mortgage loan, the time remaining in the loan, and the volatility of the real-estate market.

It is not possible to find a closed-form solution of the problem that gives the value of the portfolio at any instant. Dwyne and Wilmott (3) calculated the value of an American put with a variable exercise price in discrete time.

Our approach is as follows. Calculate the value of the portfolio every month using numerical methods. If the value of the portfolio is negative, the homeowner should default. This is the optimal strategy for the homeowner. If the value of the portfolio is positive, the borrower should not default on the loan, otherwise he will give up something of positive value in exchange for nothing.

### 3. Value of the Put

We use the binomial option-pricing model of Cox, Ross and Rubinstein (2) and follow the procedure outlined in Hull (4) to find the value of the put.

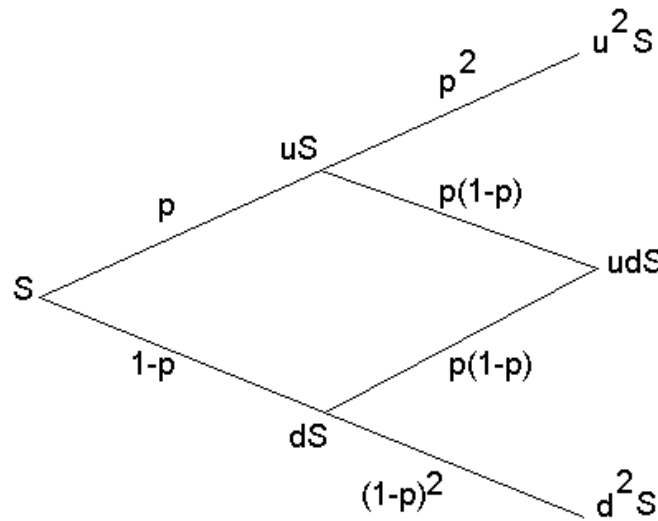


Fig. 3: Two-period binomial option pricing model with parameters  $u$ ,  $d$  and  $p$ .

Divide the life of an American put into  $N$  subintervals of length  $\Delta t$ . Draw a lattice diagram as shown in Figure 3. Assume that the value of the asset will move upward in a given interval with the probability  $p$ . The upward shift will be equivalent to the multiplication by  $u$ . Likewise, the asset can move downward in an interval with probability  $(1 - p)$  with a multiplicative factor  $d$ . Because the branches of the tree recombine, the resulting diagram is a lattice diagram.

We can extend the diagram to  $N$  periods. Count the nodes from  $0 \leq i \leq N$  and  $0 \leq j \leq i$ . Define the node  $(i, j)$  as  $i$ th node from left to right, and  $j$ th from bottom to top. The price of the underlying asset, the house, at the  $(i, j)$  node is  $H_0 u^j d^{i-j}$ . The value of a put at expiration time,  $N$  months, is

$$P_N = \max(X - H_N, 0), \quad (3.1)$$

where  $H_N$  is the value of the house after  $N$  months. Thus the value of the put at any of the nodes at expiration is

$$P_{N,j} = \max(X - H_0 u^j d^{N-j}, 0), \text{ for } j = 0, 1, 2, \dots, N \quad (3.2)$$

There is a probability  $p$  of moving from node  $(i, j)$  at the time  $i\Delta t$  to the node  $(i+1, j+1)$  at the time  $(i+1)\Delta t$ . Similarly, there is a probability  $(1 - p)$  of moving from node  $(i, j)$  at the time  $i\Delta t$  to the node  $(i+1, j)$  at the time  $(i+1)\Delta t$ . Suppose there is *no early exercise* of the put, then the risk-neutral evaluation gives

$$P_{i,j} = e^{-r\Delta t} [pP_{i+1,j+1} + (1-p)P_{i+1,j}], \text{ for } 0 \leq i \leq N-1 \text{ and } 0 \leq j \leq i \quad (3.3)$$

With early exercise, the above value must also be compared with the intrinsic value of the put at time  $i\Delta t$ . This gives the value of the American put as

$$P_{i,j} = \max\{X_i - H_0 u^j d^{i-j}, e^{-r\Delta t} [pP_{i+1,j+1} + (1-p)P_{i+1,j}]\}, \text{ for } 0 \leq i \leq N-1 \text{ and } 0 \leq j \leq i \quad (3.4)$$

Equation (3.4) gives the value of an American put option in discrete time.

As pointed out by Cox, Ross and Rubinstein, binomial option-pricing formula converges to the Black-Scholes model in the continuous-time framework. To affect the change, one has to make the following substitutions

$$u = e^{\sigma\sqrt{\Delta t}} \quad (3.5)$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}} \quad (3.6)$$

$$p = \frac{e^{r\Delta t} - d}{u - d} \quad (3.7)$$

This means that instead of calculating three separate variables,  $u$ ,  $d$ , and  $p$ , one needs only one variable, namely  $\sigma$ , to capture the stochastic behavior of the asset. Further, we have to assume that the asset follows the Brownian motion.

#### 4. Numerical Results

One can use the Maple code in the Appendix to find the value of the put and the value of the portfolio (2.2). The portfolio value is the house value minus the balance on the mortgage loan plus the value of the option to default.

Suppose a person buys a \$100,000 house by making a 20% down payment, and takes a 30-year mortgage at 6% annual interest rate. His mortgage loan is \$80,000 initially. Assume that the risk-free rate is 3% per year and the  $\sigma$  of the house is 40%. His down payment is \$20,000. The calculated value of the put option is \$22,333. The homebuyer has paid only \$20,000 to acquire a portfolio worth \$42,333!

The homeowner cannot cash in the portfolio. If he exercises the put option, he will lose the down payment and the house. The bank will take over the house to satisfy the \$80,000 balance on the loan.

Banks are willing to give this put away because they are confident that the homeowner will not exercise it. The value of the put decreases due to two factors: the time remaining in the loan decreases and the declining balance of the loan. The banks reduce their risk exposure in several ways. They can require a higher down payment, which discourages the homeowner to exercise the option. They can charge "points" on the mortgage loan, or simply charge a higher interest rate on the loan. The banks are usually not concerned about the possibility of a decrease in home values.

A put is akin to an insurance policy. It can preserve the value of an asset. Investors buy protective puts on stocks, or on stock indices, to safeguard their stock holdings against a sudden downturn in the stock market. The banks provide this type of insurance to the homeowners in case the home prices fall substantially.

Consider the possibility of a drop in the house values. In our example, the price of the house drops from \$100,000 to \$70,000 in five years. This translates into the annual rate of 7.394% during this time. The balance of the loan is \$74,443. The house is "under water." Should the homeowner satisfy the loan obligation by giving up the house with a lower value? The answer is *no* because the total value of the portfolio is \$19,108, including the put option worth \$23,551.

The following table gives the numerical values five years after the purchase of the house. The time remaining in the mortgage is 25 years and the balance in the loan is \$74,443.

House value	Put value	Portfolio value
\$70,000	\$23,551	\$19,108
60,000	26,689	12,245
50,000	30,725	6,281
40,000	36,259	1,815
35,000	39,922	479
31,000	43,443	0

Suppose a homeowner decides to default when the price of the house has fallen to \$60,000. He will lose the house, which provides him shelter, and lose \$12,245 in the value of the portfolio. He may use the loan payments of \$479.64 to rent an equivalent house.

The bank may decide to hold on to the house until the price rises to \$74,443 and then sell it. Or it may simply sell the house and take a loss.

## Appendix

```
#Real estate problem
#Find the value of the option to default on a mortgage loan
#Calculate the parameters
#Define, N = number of months remaining in the mortgage
#T = initial term of the mortgage in months
#S = market value of the asset (house) at present
#B0 = initial value of the mortgage loan
#t = length of each time interval = one month
#sigma = the "volatility" of the house
#r = risk-free interest rate
#m = mortgage interest rate, per month
N:=360; T:=360; S:=100; B0:=80; t:=1./12; sigma:=.4; r:=.03; m:=.06/12;
u:=exp(sigma*sqrt(t)); d:=1/u;
a:=exp(r*t); p:=(a-d)/(u-d);
#Set up an array, Put, to calculate put values
Put:=array(1..N+1,1..N+1);
#Set the numbers in the last column = 0
for i to N+1 do Put[i,N+1]:=0 od;
#Show puts in the remaining spaces
for k to N do
for j to N+1-k do
X:=B0*((1+m)^T-(1+m)^(T-k))/((1+m)^T-1);
Put[j,N+1-k]:=max(max(X-S*u^(N+1-k-j)*d^(j-1),0), (p*Put[j,N+2-k]+(1-p)*Put[j+1,N+2-k])*exp(-r*t));
od od;
put = Put[1,1];
port=S-X+Put[1,1];
```

## References

1. Black, Fischer; Myron Scholes (1973). "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (3): 637–654.
2. Cox, J. C., S. A. Ross and M. Rubinstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7 (1979), 229-263.
3. Dewynne, J. N. and P. Wilmott, "A Note On American Options With Varying Exercise Price," *J. Austral. Math. Soc. Ser. B* 37(1995), 45-57.
4. Hull, John C., *Options, Futures, and Other Derivatives*, 8th Edition, (2012), Prentice Hall, New York.