1. MAPLE

Objective: After reading this chapter, you will solve mathematical problems using Maple

1.1   Maple

Maple is an extremely powerful program, which can be used to work out many different types of mathematical problems. Maple can do arithmetic, algebra, and calculus. It can perform summation of series, curve plotting, and many other functions. It can solve algebraic equations and differential equations. It is possible to add text in the plots, draw three-dimensional or animated plots, or draw plots in color. All plots in this book are drawn with the help of Maple. Being a versatile analytical tool, Maple has extensive applications in science, mathematics, engineering, and finance. Any time spent in learning this program can pay rich dividends in higher productivity.

Since Maple interprets capital and lower case letters distinctly, we should use the symbols carefully. Maple has many built in mathematical functions and constants, such as

<p style="text-align:center"><strong style="color:red">ln, exp, Pi, sin, sqrt</p>

An extensive on-line help facility can guide the beginner. Maple can do exact arithmetic calculations and displays the answer in its totality. For example, we may need the exact value of 2^{64}, or the factorial of 50, or the value of π to 50 significant figures. We can do it as follows, ending each line with a semicolon, and striking the return key.

2^64;

$$18446744073709551616$$

50!;

$$30414093201713378043612608166064768844377641568960512000000000000$$

evalf(Pi,50);

$$3.1415926535897932384626433832795028841971693993751$$

Here

<p style="text-align:center"><strong style="color:red">evalf</p>

calculates the result in floating point with 50 significant figures. Maple can also do algebraic calculations. For instance, to solve the equations

$$5x + 6y = 7$$
$$6x + 7y = 8$$

for x and y, we enter the instructions as follows:

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```
eq1:=5*x+6*y=7;
```

$$eq1 := 5\,x + 6\,y = 7$$

```
eq2:=6*x+7*y=8;
```

$$eq2 := 6\,x + 7\,y = 8$$

```
solve({eq1,eq2},{x,y});
```

$$\{y = 2,\, x = -1\}$$

The symbol

```
:=
```

is used specifically to *define* objects in Maple. In other words, if we type in

```
eq1;
```

then the computer will recall the equation defined as `eq1` and display

$$5\,x + 6\,y = 7$$

Maple can also do differentiation and integration. Consider the function

$$x^3 + \frac{\ln x}{x}$$

To differentiate this function with respect to x, we type in

```
diff(x^3+ln(x)/x,x);
```

$$3\,x^2 + \frac{1}{x^2} - \frac{\ln(x)}{x^2}$$

To integrate the result with respect to x, recreating the original function, we enter

```
int(%,x);
```

$$x^3 + \frac{\ln x}{x}$$

Here we use `%` as a symbol to designate the previous expression.

Examples

1.1. Find the value of a bond that pays interest continuously at the rate of p dollars per year. The continuously compounded discount rate for such a bond is r. The bond will mature after T years and will pay face amount F dollars.

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This is an elementary problem in continuous time finance. In discrete time, the relationship between the present value P, and the future value F, of a sum of money is

$$F = P(1 + r)^T \tag{1.1}$$

where r is the annually compounded interest rate, and T is the time in years. The same relation in continuous time is

$$F = Pe^{rT} \tag{1.2}$$

We calculate the present value as

$$P = Fe^{-rT} \tag{1.3}$$

Consider a small time interval dt, between time t and $t + dt$. The interest produced during this interval is pdt. The present value of this interest payment is obtained by multiplying by the discount factor e^{-rT}, that is, $pe^{-rT}dt$. The total present value of the interest payments is found by integrating the discounted value over time to maturity. Then we add to it the discounted value of the face amount F available after time T to get the total present value of the bond. This is set up as

$$B = \int_0^T pe^{-rt}\, dt + Fe^{-rT} \tag{1.4}$$

To find the result of integration by using Maple, we key in

```
int(p*exp(-r*t),t=0..T)+F*exp(-r*T);
simplify(%);
B:=%;
```

After some simplification, we get the result as

$$B = p(1 - e^{-rT})/r + Fe^{-rT} \tag{1.5}$$

For a zero-coupon bond, $p = 0$, and (1.4) becomes

Zero-coupon bond $$B = Fe^{-rT} \tag{1.6}$$

For a perpetual bond, $T = \infty$, $e^{-rT} = 0$, and (1.4) becomes

Perpetual bond $$B = p/r \tag{1.7}$$

Consider a bond with coupon 8%, paid continuously, maturing after 10 years, when the investors require a return of 6%. To find a numerical answer, substitute $p = \$80$ per year, $F = \$1000$, $T = 10$ years, and $r = 6\%$. We may do this by entering

```
subs(p=80,F=1000,T=10,r=.06,B);
evalf(%);
```

In this case   **subs**

_____

is used to substitute the numerical values in *B*. The result is $1150.40. ♥

The value of this bond at maturity is just its face value, $1000. Thus over the next ten years, the value of this bond will actually drop gradually to its face amount. This drop, however, is not linear. We can see a picture of this process by using the graphics capability of Maple. We can do it as follows.
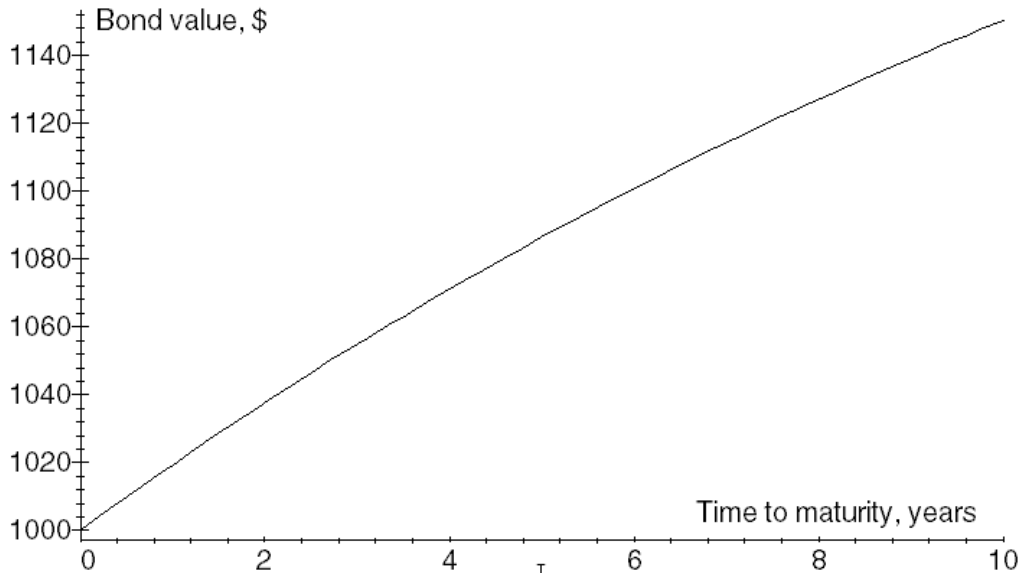
```
subs(p=80,F=1000,r=.06,B);
plot(%,T=0..10);
```


Fig. 1.1: Change in the value of a bond with passage of time

On closer inspection, we see the non-linearity in the resulting curve.

1.2. Suppose you deposit $300 at the beginning of every month in an account that pays 6% annual interest, compounding it monthly. What is the total amount in this account after 25 years?

On a monthly basis, the interest rate is ½%, or .005. There are 25*12 = 300 months in 25 years. The first deposit will become $300(1.005)^{300}$ after 300 months, the second one will become $300(1.005)^{299}$, and the last one will be $300(1.005)$. We can add them up as a series:

$$S = 300(1.005)^{300} + 300(1.005)^{299} + \ldots + 300(1.005)$$

There are 300 terms in this series. We can write summation symbolically as

$$S = \sum_{i=1}^{300} 300(1.005)^{i}$$

We can do the summation with Maple as

_____

```
sum(300*1.005^I,i=1..300);
```

The answer comes out to be $208,937.68. ♥

1.3. The credit card debt of a family is growing at a rate proportional to the debt. The debt in 2005 was $10,000, and in 2010, it is $15,000. Find the debt in 2011.

Suppose the debt at any time is $B(t)$, then rate of growth of debt is dB/dt. If the rate of growth of debt is proportional to debt, we can write this as

$$\frac{dB(t)}{dt} \propto B(t)$$

We can change this proportionality into an equation by multiplying by a constant. That is

$$\frac{dB(t)}{dt} = kB(t)$$

We use the following instructions to solve it with Maple

```
diff(B(t),t)=k*B(t);
dsolve(%,B(t));
```

With a slight change in notation, the result is $B(t) = C\ e^{kt}$

We notice that the debt is growing exponentially. To find the numerical value of the two constants C and k, we substitute the data for 2005 and 2010. To simplify the dates, assume $t = 0$ in 2005, $t = 5$ in 2010, and $t = 6$ in 2011. This gives us two equations,

Debt in 2005: $10,000 = C\ e^{0k}$

Debt in 2010: $15,000 = C\ e^{5k}$

The first equation gives $C = 10,000$. Putting it in the second equation,

$$15,000 = 10,000\ e^{5k}$$

Or, $e^{5k} = 1.5$;   Taking logs, $5k = \ln(1.5)$;   Or, $k = (1/5)\ln(1.5)$

The general debt equation is $B(t) = 10,000 \exp[(1/5)\ln(1.5)\ t]$

To find the debt in 2011, we let $t = 6$ in the above equation. We key in

```
B=10000*exp(1/5*ln(1.5)*6);
```

The result is $16,267. ♥