1. ANALYTICAL TOOLS

Objectives: After reading this chapter, you will be able to
1. Solve linear and quadratic equations, system of linear equations
2. Use geometric series in financial calculations
3. Understand the basic concepts of statistics
4. Use Excel, Maple to solve mathematical problems
5. Understand the concept of optimization

Before we actually start studying finance and the financial management as a discipline, it is worthwhile to review some of the fundamental concepts in mathematics first. This will help us appreciate the usefulness of analytical techniques as powerful tools in financial decision-making. We shall briefly review elementary algebra, basic concepts in statistics, and finally learn Excel or Maple as a handy way to cut through the mathematical details.

Our approach toward learning finance is to translate a word problem into a mathematical equation with some unknown quantity, solve the equation and get the answer. This will help us determine an exact answer, rather than just an approximation. This will lead to a better decision.

1.1 Linear Equations

To review the basic concepts of algebra, we look at the simplest equations first, the linear equations. These equations do not have any squares, square roots, or trigonometric or other complicated mathematical functions.

Example

1.0. Suppose John buys 300 shares of AT&T stock at $26 a share and pays a commission of $10. When he sells the stock, he will have to pay $10 in commission again. Find the selling price of the stock, so that after paying all transaction costs, John's profit is $200.

Let us define profit π as the difference between the final payoff F, after commissions, and the initial investment I_0, including commissions. We can write it as a linear equation as follows

$$\pi = F - I_0$$

We require a profit of $200, thus, $\pi = 200$. Suppose the final selling price of the stock per share is x, the number we want to calculate. Selling 300 shares at x dollars each, and paying a commission of $10, gives the final payoff as, $F = 300x - 10$. The initial investment in the stock, including commission, is $I_0 = 300(26) + 10 = \$7810$. Make these substitutions in the above equation to obtain

$$200 = 300x - 10 - 7810$$

Moving things around, we get $200 + 10 + 7810 = 300x$

Or, $$8020 = 300x$$

Or, $$x = 8020/300 = 26.73333333 \approx \$26.73$$ ♥

This means that the stock price should rise to $26.73 to get the desired profit. Note that the answer has a dollar sign and it is truncated to a reasonable degree of accuracy, namely, to the nearest penny.

Consider another problem involving dollars, doughnuts, and coffee.

1.1. Jane works in a coffee shop. During the first half-hour, she sold 12 cups of coffee and 6 doughnuts, and collected $33 in sales. In the next hour, she served 17 cups of coffee and sold 8 doughnuts, for which she received $46. Find the price of a cup of coffee and that of a doughnut.

This is an example where we have to find the value of *two* unknown quantities. The general rule is that you need two equations to find two unknowns. We have to develop two equations by looking at the sales in the first half-hour and in the second hour. Suppose the price of a cup of coffee is x dollars, and that of a doughnut y dollars.

First half-hour, 12 cups and 6 doughnuts for $33, gives $12x + 6y = 33$
Second hour, 17 cups and 8 doughnuts for $46, gives $17x + 8y = 46$

Now we have to solve the above equations for x and y.

First, try to eliminate one of the variables, say y. You can do this by multiplying the first equation by 8 and the second one by 6, and then subtracting the second equation from the first. This gives

$$8*12x + 8*6y = 8*33$$
$$6*17x + 6*8y = 6*46$$

Subtracting second from first, $(8*12 - 6*17)x = 8*33 - 6*46$

Simplifying it, $-6x = -12$, or $x = 2$ ♥

Substituting this value of x in the first equation, we have $12*2 + 6y = 33$

Or, $$6y = 33 - 24 = 9$$

$$y = 9/6 = 3/2$$ ♥

The answer is that a cup of coffee sells for $2 and a doughnut for $1.50.

1.2 Non-linear Equations

Non-linear equations contain higher powers of the unknown variable, or the variable itself may show up in the power of a number. For instance, a quadratic equation is a non-linear equation. The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \tag{1.1}$$

The roots of this equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.2}$$

Consider the following examples of non-linear equations.

Examples

1.2. Solve for x: $1.113^x = 2.678$

First, we recall the basic property of logarithm functions, namely,

$$\ln(a^x) = x \ln a$$

Taking the logarithm on both sides of the given equation, we obtain

$$x \ln(1.113) = \ln(2.678)$$

Or, $$x = \frac{\ln(2.678)}{\ln(1.113)} = \frac{0.9850702}{0.1070591} = 9.201$$ ♥

1.3. Solve for x, $(2 + x)^{2.11} = 16.55$

This gives $2 + x = (16.55)^{1/2.11}$

Or, $x = (16.55)^{1/2.11} - 2 = 1.781$ ♥

1.4. Find the roots of $5x^2 + 6x - 11 = 0$

Here $a = 5$, $b = 6$, and $c = -11$. This gives us

$$x = \frac{-6 \pm \sqrt{36 - 4(5)(-11)}}{10} = \frac{-6 \pm \sqrt{256}}{10} = \frac{-6 \pm 16}{10} = -\frac{11}{5} \text{ or } 1$$ ♥

1.3 Geometric Series

In certain problems in financial management, we have to deal with a series of cash flows. When we look at the present value, or the future value, of these cash flows, the resulting series is a geometric series. Thus, geometric series will play an important role in managing money. Let us consider a series of numbers represented by the following sequence

$$a\ ,\ ax\ ,\ ax^2\ ,\ ax^3\ ,\ \ldots\ ,\ ax^{n-1}$$

The sequence has the property that each number is multiplied by x to generate the next number in the list. There are altogether n terms in this series, the first one has no x, the second one has an x, and the third one has x^2. By this reasoning, we know that the nth term must have x^{n-1} in it. This type of series is called a *geometric series*. Our concern is to find the sum of such a series having n terms with the general form

$$S = a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} \tag{1.3}$$

To evaluate the sum, proceed as follows. Multiply each term by x and write the terms on the right side of the equation one-step to the right of their original position. We can set up the original and the new series as follows:

$$S = a + ax + ax^2 + ax^3 + \ldots + ax^{n-1}$$

$$xS = \qquad ax + ax^2 + ax^3 + \ldots + ax^{n-1} + ax^n$$

If we subtract the second equation from the first one, most of the terms will cancel out, and we get

$$S - xS = a - ax^n$$

Or,

$$S(1 - x) = a(1 - x^n)$$

or,

$$S_n = \frac{a\,(1 - x^n)}{1 - x} \tag{1.4}$$

This is the general expression for the summation of a geometric series with n terms, the first term being a, and the ratio between the terms being x. This is a useful formula, which we can use for the summation of an annuity.

If the number of terms in an annuity is infinite, it becomes a *perpetuity*. To find the sum of an infinite series, we note that when n approaches infinity, $x^n = 0$ for $x < 1$. Thus, the sum for an infinite geometric series becomes

$$S_\infty = \frac{a}{1 - x} \tag{1.5}$$

Examples

1.5. Find the sum of $3 + 6 + 12 + 24 \ldots$, 13 terms

We identify $a = 3$, $x = 2$, and $n = 13$. Putting these in (1.4), we get

$$S = \frac{a\,(1 - x^n)}{1 - x} = \frac{3\,(1 - 2^{13})}{1 - 2} = 3(2^{13} - 1) = 24{,}573 \ ♥$$

1.6. Find the sum of $1.7 + 2.21 + 2.873 \ldots$, 11 terms

Here $a = 1.7$, and $x = 2.21/1.7 = 1.3$. Also, $n = 11$. This gives

$$S = \frac{a\,(1 - x^n)}{1 - x} = \frac{1.7\,(1 - 1.3^{11})}{1 - 1.3} = \frac{1.7(1.3^{11} - 1)}{1.3 - 1} = 95.89 \ ♥$$

1.7. Find the value of $\sum_{i=1}^{100} \frac{25}{1.12^i}$

The mathematical expression possibly means a sum of one hundred annual payments of \$25 each, discounted at the rate of 12% per annum. Write it as

$$\sum_{i=1}^{100} \frac{25}{1.12^i} = \frac{25}{1.12} + \frac{25}{1.12^2} + \frac{25}{1.12^3} + \frac{25}{1.12^4} + \ldots + \frac{25}{1.12^{100}}$$

This is a geometric series, with the initial term $a = \frac{25}{1.12}$, the multiplicative factor $x = \frac{1}{1.12}$, and the number of terms, $n = 100$. Use the equation

$$S_n = \frac{a\,(1 - x^n)}{1 - x} \qquad (1.4)$$

to get

$$S_n = \frac{(25/1.12)\,(1 - 1/1.12^{100})}{1 - 1/1.12} = 208.33 \ ♥$$

The keystrokes needed to perform the calculation on a TI-30X calculator are as follows:

25 [÷] 1.12 [×][(] 1 [−] 1 [÷] 1.12 [y^x] 100 [)][÷][(] 1 [−] 1 [÷] 1.12 [)][=]

The Maple instruction is as follows:

```
sum(25/1.12^i,i=1..100);
```

1.4 Video 01B, Elements of Statistics

Probability theory plays an important role in financial planning, forecasting, and control. At this point, we shall briefly review some of the basic concepts of probability and statistics. In many instances, we have to deal with quantities that are not known with certainty. For example, what is the price of IBM stock next year or the temperature in Scranton tomorrow? The future is unpredictable. The market may go up tomorrow, or down. One way to get a handle on the unknown is to describe it in terms of probabilities.

For instance, there is a 30% chance that it may rain tomorrow. On the other hand, there is an even chance that the market may go up or down on a given day. The sum of the probabilities for all possible outcomes is, of course, one.

We may base the probabilities of different outcomes on the past observations of a certain event. For instance, we look at the stock market for the last 300 trading days and we notice that on 156 days it went up. Then it is fair to say that it has a 156/300 = 0.52 = 52% chance that it may go up tomorrow as well. A complete set of all probabilities is a probability distribution. The probability distribution for the stock market may look like this:

Outcome	**Probability**
Market moves up	52%
Market moves down	48%

In the above case, we are assuming that the market does not end up exactly at the closing level of the previous day.

The distribution in the previous example is a discrete probability distribution. Another example of such a distribution is the set of probabilities for the outcomes of a roll of dice. With a single die, the probability is 1/6 each of getting a 1, or 2, or 3, and so on.

A probability distribution may be continuous, such as the normal probability distribution. The probability distribution describing the life expectancy of human beings, or machines, is a continuous distribution. At present, we shall try to describe the uncertainty in terms of discrete probability. We are going to use a subjective probability distribution to describe the uncertain future.

We can find the expected value of a certain quantity by multiplying the probability of each outcome by the value of that outcome.

Example

1.8. A project has the following expected cash flows

State of the Economy	***Probability***	***Cash Flow X***
Good	60%	$10,000
Fair	30%	$6,000
Poor	10%	$2,000

To find the expected cash flow, we compute

$$E(X) = .6*\$10,000 + .3*\$6,000 + .1*\$2,000 = \$8,000$$

Consider a random variable X. Its outcome is X_1 with a probability P_1, X_2 with a probability P_2, and so on. In general, the outcome is X_i with a probability P_i. Then the expected value of X is

$$E(X) = P_1X_1 + P_2X_2 + ... + P_iX_i$$

We may write this as

Expected value of X, $$E(X) = \sum_{i=1}^{n} P_iX_i = \overline{X} \quad (1.6)$$

Next, we would like to know how much scatter, or dispersion, is present in this expected value of X. We may estimate this by the variance of X, or the standard deviation of X, defining them as follows.

Variance of X, $$\text{var}(X) = \sum_{i=1}^{n} P_i(X_i - \overline{X})^2 \quad (1.7)$$

Standard deviation of X, $$\sigma(X) = \sqrt{\text{var}(X)} \quad (1.8)$$

In the above example the standard deviation of the cash flow is

$$\sigma(X) = \sqrt{.6*(10,000 - 8000)^2 + .3*(6000 - 8000)^2 + .1*(2000 - 8000)^2} = \$2683$$

This figure represents the uncertainty, or the margin of error, in the cash flow.

At times, it is necessary to find the mutual dependence of two different events. For example, we start two separate projects X and Y. The following table shows their expected cash flows. The first project X, is the same as the one discussed above on the previous page.

State of the Economy	Probability	Cash Flow X	Cash Flow Y
Good	60%	$10,000	$12,000
Fair	30%	$6,000	$8,000
Poor	10%	$2,000	$6,000

The two projects seem to be in step, both making more money in good economy and less in poor economy. They seem to have a close relationship. Is there a way to measure it quantitatively? The answer is yes, by using a measure called the correlation coefficient. First we define the covariance between two random variables X and Y as the

Covariance between X and Y, $$\text{cov}(X,Y) = \sum_{i=1}^{n} P_i(X_i - \overline{X})(Y_i - \overline{Y}) \quad (1.9)$$

To find the covariance between the two projects, we must first find the expected value of Y. Do it as

$$\overline{Y} = .6*\$12,000 + .3*\$8,000 + .1*\$6,000 = \$10,200$$

Next, we find

$$\text{cov}(X,Y) = .6*(10,000 - 8000)*(12,000 - 10,200) + .3*(6,000 - 8,000)*(8,000 - 10,200) + .1*(2,000 - 8,000)*(6,000 - 10,200)$$

$$= 6,000,000$$

The six-million figure found above is not particularly meaningful. We next introduce a more practical measure of interdependence of two projects, the correlation coefficient, defined as

$$r(X,Y) = \frac{\text{cov}(X,Y)}{\sigma(X)\sigma(Y)} \tag{1.10}$$

Write the above equation as

$$\text{cov}(X,Y) = r(X,Y)\sigma(X)\sigma(Y) \tag{1.11}$$

We already know $\sigma(X)$ to be \$2683. We also evaluate $\sigma(Y)$ to be

$$\sigma(Y) = \sqrt{.6(12,000 - 10,200)^2 + .3(8000 - 10,200)^2 + .1(6000 - 10,200)^2} = \$2272$$

The smaller value of $\sigma(Y)$ indicates that the cash flows are more tightly bunched. Finally, we find the correlation coefficient as

$$r(X,Y) = \frac{\text{cov}(X,Y)}{\sigma(X)\sigma(Y)} = \frac{6,000,000}{2683*2272} = .9843 \ ♥$$

Note that $r(X,Y)$ is a pure number and its value always lies between +1 and −1. That is

$$-1 < r(X,Y) < 1 \tag{1.11}$$

If the two projects are completely (meaning 100%), positively (meaning, moving in the same direction) correlated, the correlation coefficient between them is +1. This will be the case if one project is a carbon copy of the other one. If they are totally unrelated, the coefficient should be 0. This will be the case if one project is completely independent of the other one. If the two projects are such that whatever happens in one, the exact opposite happens with the other, then their correlation coefficient is −1.

The high value of $r(X,Y)$, .9843, in the above example is not particularly surprising because the two projects go hand in hand, performing well in good times and poorly in bad times. Some of these ideas are particularly helpful in understanding the risk and return of different portfolios.

1.5 Excel

It is important that the students are able to set up finance problems using Excel, which is now a standard of business and industry. A good working knowledge of this software should be an integral part of every business student's education. Almost all business programs offer courses in the use of this software. If you want to brush up your skill in the use of Excel, you may go the following Microsoft website for a variety of tutorials.

http://office.microsoft.com/en-us/training/CR100479681033.aspx

To get started on Excel, consider one of the previous problems that we solved by using the logarithm function.

1.2. Solve for x: $1.113^x = 2.678$

Set up the table shown below. Adjust the number in the green cell B2 until the numbers in cells B3 and B4 come very close together. B2 gives the answer.

	A	B
1	Base =	1.113
2	Unknown power =	9.201184226
3	Result (given) =	2.678
4	Result(calculated) =	=B1^B2

Next, consider example on page 7 again. Set it up on Excel as follows. The numerical results of the formulas in cells B5:B10 are given in green cells C5:C10. The principal advantage of Excel is that it can handle large tables of numbers.

	A	B	C	D
1	State of the Economy	Probability	Cash Flow X	Cash Flow Y
2	Good	60%	10000	12000
3	Fair	30%	6000	8000
4	Poor	10%	2000	6000
5	E(X)	=B2*C2+B3*C3+B4*C4	8000	
6	E(Y)	=B2*D2+B3*D3+B4*D4	10200	
7	Cov(X,Y)	=B2*(C2-B5)*(D2-B6)+B3*(C3-B5)*(D3-B6)+B4*(C4-B5)*(D4-B6)	6000000	
8	sigma(X)	=SQRT(B2*(C2-B5)^2+B3*(C3-B5)^2+B4*(C4-B5)^2)	2683.28157	
9	sigma(Y)	=SQRT(B2*(D2-B6)^2+B3*(D3-B6)^2+B4*(D4-B6)^2)	2271.56334	
10	r(X,Y)	=B7/B8/B9	0.98437404	

1.6 Video 01C, Maple

Available on desktop computers, Maple is an extremely powerful analytical tool. Working with Maple is quite easy. The help facility in Maple is very valuable and it can guide the user through various steps, using plenty of examples. Maple has extensive application in science, mathematics, engineering, and finance. Time spent in learning this program can pay rich dividends in terms of greater accuracy and higher productivity. The following instructions will get you started with Maple.

Since Maple interprets capital and lower case letters distinctly, we should use the symbols carefully. Maple has many built in mathematical functions and constants, such as

```
ln, exp, Pi, sin, sqrt
```

Maple can do exact arithmetic calculations and displays the answer in its totality. For example, we need the exact value of 2^{64}, or the factorial of 50, or the value of π to 50 significant figures. We do this as follows: enter the commands at the > prompt, end each line with a semicolon, and strike the return key.

```
2^64;
```

18446744073709551616

```
50!;
```

30414093201713378043612608166064768844377641568960512000000000000

```
evalf(Pi,50);
```

3.1415926535897932384626433832795028841971693993751

Here

```
evalf
```

calculates the result in floating point with 50 significant figures. Maple can also do algebraic calculations. For instance, to solve the equations

$$5x + 6y = 7$$
$$6x + 7y = 8$$

for x and y, enter the instructions as follows:

```
eq1:=5*x+6*y=7;
```

$$eq1 := 5\,x + 6\,y = 7$$

```
eq2:=6*x+7*y=8;
```

$$eq2 := 6\,x + 7\,y = 8$$

```
solve({eq1,eq2},{x,y});
```

$$\{y = 2, x = -1\}$$

The symbol `:=` is used specifically to *define* objects in Maple. In other words, if we type

```
eq1;
```

then the computer will recall the equation defined as `eq1` and display it as

$$5\,x + 6\,y = 7$$

Maple can also do differentiation and integration. Consider the function

$$x^3 + \frac{\ln x}{x}$$

To differentiate this function with respect to *x*, we type in

```
diff(x^3+ln(x)/x,x);
```

$$3\,x^2 + \frac{1}{x^2} - \frac{\ln(x)}{x^2}$$

To integrate the result with respect to *x*, recreating the original function, we enter

```
int(%,x);
```

$$x^3 + \frac{\ln x}{x}$$

Here we use `%` as a symbol to designate the previous expression.

We can also use Maple to plot functions. For instance, if we want to see the visual representation of the well-known sine wave, we write

```
plot(sin(x),x=0..2*Pi);
```

which gives the diagram shown in Figure 1.1.

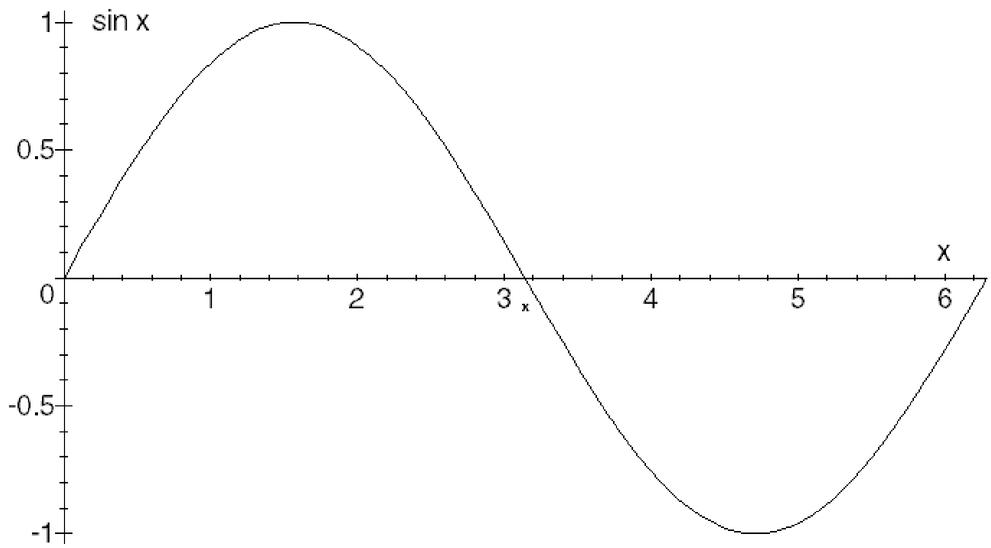


Fig. 1.1: Plot of sin *x* for $0 < x < 2\pi$

It is possible to add text in the plots, draw three-dimensional or animated plots, and draw plots in color. All plots in this book are drawn with the help of Maple.

1.7 Wolfram|Alpha

Another mathematical software, Mathematica, has capabilities similar to Maple. It can also perform all the mathematical problems equally well. Mathematica has a website at Wolfram|Alpha, which is free to use. The instructions at Wolfram|Alpha are almost identical to those in Maple. You should explore this website and use it when you do not have access to Maple.

For instance, to solve the equations

$$5x + 6y = 7$$
$$6x + 7y = 8$$

for x and y, enter the instructions as follows:

```
5x+6y=7,6x+7y=8
```

1.8 Optimization

When we are managing any operation, we should look for an optimal value of its different parameters. Consider the example of a large aircraft flying from New York to London. What is the optimal amount of fuel it should carry? If it carries too much fuel, it carries too much load, and thus it will cost too much to fly. If it carries too little fuel, it may hit the drink. It should carry enough fuel to fly across the Atlantic, and have enough in reserve for any emergencies.

Next, consider an example from corporate finance. What should be the optimal mix of debt and equity for a firm? If the firm has too much equity, it is not taking advantage of the cheaper form of capital, namely, debt. The debt also gives it, what is known as the *tax shield*, an additional value due to the interest being a tax-deductible expense. If the company has too much debt, it may become too risky and head towards bankruptcy. The goal is to keep the balance between the positive features of debt (tax shield) against its negative features (bankruptcy costs). At the optimal point, these two factors cancel out, and the value of the firm reaches a maximum.

We may also look at the *optimization* phenomenon from a different angle. We know that the companies become riskier as they acquire more debt. The cost of capital for riskier companies is higher than those that are financially sound. Thus, we have to find the optimal mix of debt and equity where the weighted average cost of capital is minimized.

Let us consider one more example. This is optimal level of inventory for a firm. If the firm carries too much inventory, it is being wasteful. The inventory turnover ratio is low, and the items go out of style while sitting on the shelf. On the other hand, if it has too little inventory, then the customers cannot find what they are looking for. This can lead to serious problems in its sales, or production runs.

Frequently we have to use calculus to solve optimization problems. First, we devise a *function* that would represent the quantity that we want to optimize, for instance, the cost of keeping a certain amount of inventory on hand. Next, we have to differentiate the function with respect to a certain *variable*, and set the derivative equal to zero. Then we solve this algebraic equation for the unknown variable, which gives us the optimal value.

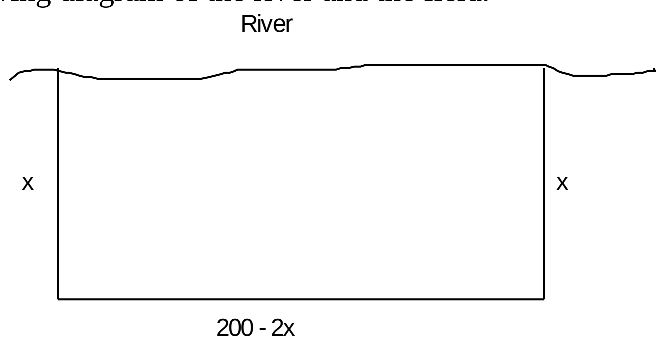
We may have to search the optimal value within the practically possible values. At times, we have to look at the second derivative to see if we are at a maximum or a minimum. We may summarize this procedure as follows.

1. Suppose the function is C, and the independent variable is x.
2. Calculate the first derivative of C, namely, $\frac{dC}{dx}$.
3. Solve $\frac{dC}{dx} = 0$ for x.
4. To check for a maximum or a minimum, calculate the second derivative, $\frac{d^2C}{dx^2}$. If the second derivative is positive at x found from (3), we have reached a minimum. If the second derivative is negative, we are at a maximum. If the second derivative is zero, then the test is inconclusive.

Examples

1.9. Adams the farmer has 200 yards of a fence that he wants to use as a rectangular enclosure. This enclosure will have only three sides, because he intends to use a river as the fourth side. What is the maximum area that he can enclose with the available fence?

Consider the following diagram of the river and the field.



Suppose Adams makes the rectangle with sides x perpendicular to the river. Then the side of the rectangle parallel to the river should be $200 - 2x$. The area of the rectangle will be $x(200 - 2x)$. Thus the total cost function is $C = x(200 - 2x)$, and the independent variable is x. Following the recipe outlined above, we proceed as follows.

$$C = 200x - 2x^2$$

$$\frac{dC}{dx} = 200 - 4x = 0$$

This gives us $x = 50$

Thus the enclosure should come out 50 yards perpendicularly from the river, and 100 yards along the river. The total area enclosed by this fence will be 50*100 = 5,000 square yards. ♥

1.10. Arthur Company is packaging soup in cans. The volume of the soup in each can is 400 cm^3. Find the dimensions of a cylindrical can that will minimize the amount of tin sheet used in the process.

Suppose the can has radius r and length l. The area of the circular ends is each πr^2 and the area of the cylindrical surface is $2\pi rl$. Thus, the total surface area of the can, or the amount of tin plate used in it is $2\pi r^2 + 2\pi rl$. This is the function we want to minimize.

The function has two variables r and l. We must eliminate one of them to proceed further. We recall that the volume of a cylinder is $\pi r^2 l$. We are given that the volume of our soup can is 400 cm^3. Thus $400 = \pi r^2 l$. This gives us

$$l = \frac{400}{\pi r^2}$$

Substituting it in the function, we get

$$C = 2\pi r^2 + 2\pi r\left(\frac{400}{\pi r^2}\right) = 2\pi r^2 + \frac{800}{r}$$

Now we seek the help of Maple to do the differentiation and solve the equation. The proper commands are

```
C:=2*Pi*r^2+800/r;
diff(C,r)=0;
solve(%);
evalf(%);
```

This is a cubic equation with three roots, two of them are imaginary. The only practical value is $r = 3.9929$ cm. This is the radius of the can. The diameter of the can is twice that much, or 7.9859 cm. To find the length of the can, we substitute $r = 3.9929$ in $\frac{400}{\pi r^2}$. Using Maple, do it as follows:

```
subs(r=3.992945422,400/Pi/r^2);
evalf(%);
```

This gives $l = 7.9859$ cm. Thus, the length of the can is equal to its diameter. ♥

1.11. The leveraged value of a corporation is related to its unleveraged value as

$$V_L = V_U + tB - b \tag{2.1}$$

Here V_L = value of the leveraged firm
t = income tax rate of the firm
V_U = value of the firm if it were unleveraged
B = amount of equity *replaced* by debt
b = bankruptcy costs of the firm

Suppose Buchanan Company has unleveraged value V_U = \$10 million, income tax rate t = .3, and bankruptcy costs $b = .005Be^{.6B}$. Find the optimal amount of debt B for this company.

According to (2.1), the function that we want to maximize is

$$C = 10 + .3B - .005Be^{.6B}$$

Using Maple, we proceed as follows:

```
C:=10+.3*B-.005*B*exp(.6*B);
diff(C,B)=0;
solve(%);
```

We get the result that the optimal level of debt for this company is B = \$4.613 million. The maximum value of the company is given by

```
subs(B=4.613070182,C);
evalf(%);
```

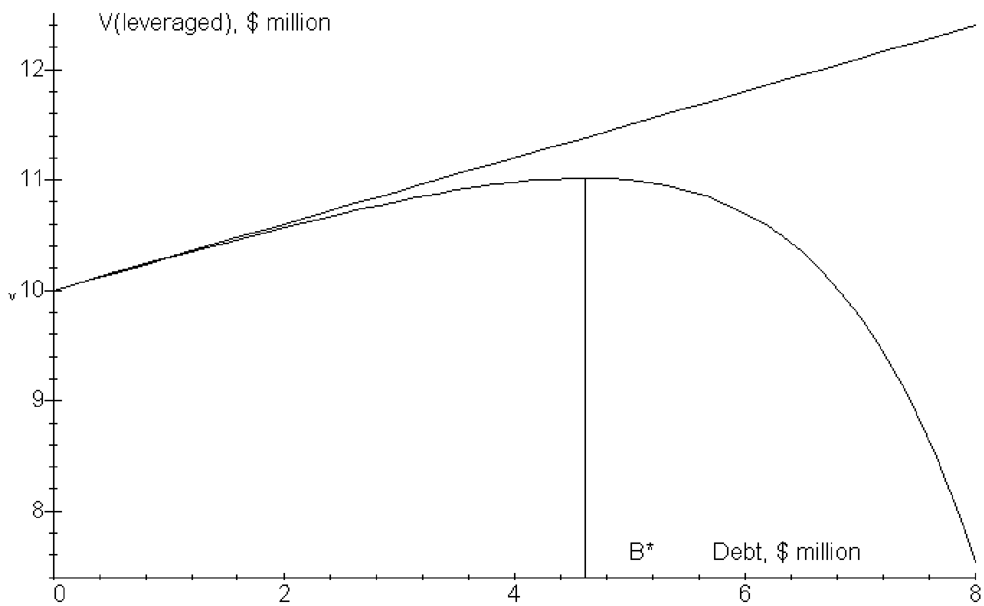


Fig. 2.1: Value maximization of a firm by using optimal debt

The maximum value of the firm is V_L = $11.017 million. This is shown in Figure 2.1.

1.9 Optimization of Working Capital

We may classify capital as being long-term or short-term. The *long-term capital* includes, stocks, bonds, preferred stock, and convertible bonds. The *short-term capital* consists of short-maturity notes, lines of credit, bank loans, and commercial paper.
In general, the companies must maintain the *working capital* at its optimal level. This means a firm should have the optimal amount of cash on hand, the optimal inventory level, and so on. The net working capital is the difference between current assets and current liabilities. One extreme point of view that the net working capital should be zero. That is not possible from a practical point of view. We should, instead, try to optimize the working capital.

Optimization is a calculus problem. Generally, we have to balance between two competing costs: the cost of *having* enough working capital and the cost of *not having* enough working capital. In other words, we should look at the cost of capital and the shortage cost.

Consider inventory that a retail store should keep on the shelves. If it has a large variety of merchandise to sell, and it keeps a large stock of each item on hand, then it has to invest a lot of money, or capital, in the inventory. Therefore, too much inventory is wasteful. Suppose, the store does not carry enough inventory then the customer cannot find what she is looking for. The store will lose not only the sale, but also the customer. Bare shelves convey the impression that the company is going out of business.

Examples

1.12. Carter Company has cost of capital 12%. This means any money invested in the inventory, or accounts receivable, is costing the company 12% per annum. The following function represents the shortage cost of the working capital

$$S = \frac{5}{x - 8} \text{ for } x > 8$$

where x is the amount invested in the working capital, in millions of dollars. Find the optimal level of working capital for Carter, and the minimum total annual cost of working capital management.

The financing cost of money invested in the working capital is $F = .12x$

The total cost is the sum of shortage cost and financing cost. In symbols, it is

$$T = S + F = \frac{5}{x - 8} + .12x$$

To find the maximum or minimum of this function, we differentiate it with respect to x, and set the derivative equal to zero. This gives us

$$\frac{dT}{dx} = -\frac{5}{(x-8)^2} + .12 = 0$$

Multiplying throughout by $(x-8)^2$, we get

$$-5 + .12(x-8)^2 = 0$$

Or, $$(x-8)^2 = 5/.12$$

Or, $$x - 8 = \pm\sqrt{5/.12}$$

$$x = 8 \pm \sqrt{5/.12} = 14.455 \text{ or } 1.545$$

Only the first value agrees with the requirement that $x > 8$. Therefore, from a practical point of view, we have just one optimal value of the working capital, namely, \$14.455 million. The total cost of working capital management will be

$$T = .12(14.455) + \frac{5}{14.455 - 8} = \$2.509 \text{ million}$$ ♥

To verify, you can do the problem in Maple as

```
T:=5/(x-8)+.12*x;
diff(T,x);
solve(%);
subs(x=max(%),T);
```

1.13. Cleveland Company's cost of capital is 15%. It has invested x dollars in current assets. The following function gives the shortage cost of current assets

$$S = 3000\, e^{-x/5000}$$

Find the optimal level of its current assets. Draw a diagram that shows the financing costs, shortage costs, total costs, and the optimal level of current assets.

The total cost, financing plus shortage cost, is represented by the function

$$T = .15x + 3000\, e^{-x/5000}$$

Differentiating it with respect to x, and setting the result equal to zero, we get

$$\frac{dT}{dx} = .15 - 3000/5000\, e^{-x/5000} = 0$$

Canceling terms, we have

$$e^{-x/5000} = .15(5/3) = .25$$

Taking natural logarithm on both sides, we get

$$-x/5000 = \ln(.25) = -1.3862944$$

Or, $x = 5000*1.3862944 = \$6931$ ♥

The shortage-cost and the financing-cost curves intersect at the point where the two costs are equal. This takes place when

$$.15x = 3000\, e^{-x/5000}$$

We can calculate the value of x by using the following Maple commands

```
solve(.15*x=3000*exp(-x/5000));
```

and the result is \$6011. We can see this in Fig. 2.2.

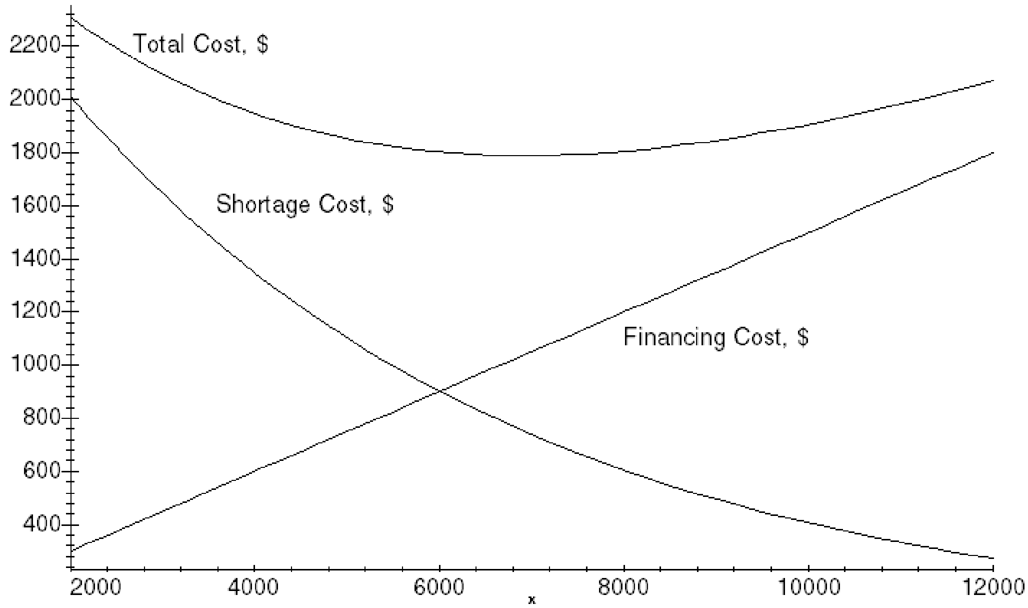


Fig. 2.2: The costs are on the scale at left, and the working capital on the scale at bottom.

Problems

Solve the following equations:

1.14. $16x - 54 = 15x - 32$ $x = 22$ ♥

1.15. $(x + 1)(x - 2) = (x - 1)(x + 2)$ $x = 0$ ♥

1.16. $(10x + 3)(3x + 4) = (5x + 6)(6x + 7)$ $x = -15/11$ ♥

1.17. $\frac{x-2}{x-3} = \frac{x-7}{x-9}$ $x = -3$ ♥

1.18. $\frac{x+4}{x+5} = \frac{x+6}{x+8}$ $x = -2$ ♥

Solve the following equations for x and y:

1.19. $2x + 6y = 32$
$5x + 8y = 45$ $x = 1, y = 5$ ♥

1.20. $3x + 4y = 15$
$5x + 8y = 45$ $x = -15, y = 15$ ♥

Solve for x,

1.21. $(1 + x)^{3.2} = 8.4$ $x = 0.9446$ ♥

1.22. $1.767^x = 3.876$ $x = 2.38$ ♥

1.23. $3.909^x = 15.99$ $x = 2.033$ ♥

Find the roots of

1.24. $2x^2 + 7x - 9 = 0$ $x = 1, -9/2$ ♥

1.25. $3x^2 + 4x - 7 = 0$ $x = 1, -7/3$ ♥

Find the sum of the following series:

1.26. 2.5 + (2.5)(.3) + (2.5)(.3)(.3) ..., infinite terms 3.571 ♥

1.27. $\frac{1}{1.1} + \frac{1}{1.1^2} + \frac{1}{1.1^3} + \ldots$ 9 terms 5.759 ♥

1.28. $\frac{30}{1.12} + \frac{30(1.05)}{1.12^2} + \frac{30(1.05)^2}{1.12^3} + \ldots$ 36 terms 386.60 ♥

1.29. $\sum_{i=1}^{10} \frac{500}{1.12^i}$ 2825.11 ♥

1.30. $\sum_{i=1}^{100} \frac{25}{1.12^i}$ 208.33 ♥

1.31. The cash flows from two projects under different states of the economy are as follows:

State of the economy	Probability	Project A	Project B
Poor	20%	\$3000	\$5000
Average	30%	\$4000	\$7000
Good	50%	\$6000	\$15,000

Find the coefficient of correlation between the two projects. .9922 ♥

1.32. The expected return from two stocks, Microsoft and Boeing, under different states of the economy are as follows:

State of the economy	Probability	Microsoft	Boeing
Poor	10%	-5%	-40%
Average	40%	10%	-10%
Good	50%	20%	30%

Find the coefficient of correlation between the two stocks. .9471 ♥

1.33. Clinton Company has cost of capital 12%. The following function represents the shortage cost for its net working capital

$$S = \frac{2}{x - 8} \quad \text{for } x > 8$$

Here S is the shortage cost in thousands of dollars, and x is the level of the net working capital, also in thousands of dollars. Find the following:

(A) The optimum level of net working capital. \$12,082 ♥

(B) The financing cost, shortage cost, and total cost at the optimal point.
\$1450, \$490, \$1940 per year ♥

1.34. Coolidge Corporation has estimated its cost of capital to be 12%, and the shortage cost of working capital as

$$S = \frac{125}{x - 25} \quad \text{for } x > 25$$

where S is the shortage cost per year in \$million, and x is the level of working capital in \$million. Find the optimal level of working capital. What is the total cost of working capital per year? \$57.27 million, \$10.75 million ♥

1.35. Eisenhower Corporation has cost of capital 10%. The shortage cost of current assets may be represented by the function $S = .3/x$, where S is the shortage cost per year (in \$million), and x is the amount invested in current assets (in \$million). Find the optimal level of, and the minimum total annual cost of, the current assets for this firm.
\$1.732 million, \$346,400 ♥

1.36. Fillmore Company has estimated its cost of capital to be 12%. The following function represents the shortage cost of its current assets,

$$S = \frac{1}{5(x - 10)} \quad \text{for } x > 10$$

where S is the shortage cost per year (in $million), and x is the amount invested in current assets (in $million). Find the optimal level of the current assets for this firm. What is the minimum annual cost of these assets? $11.291 million, $1.510 million ♥

1.37. Garfield Company's cost of capital is 12%. It has invested x (million dollars) in current assets. The shortage cost of current assets is represented by the function

$$S = 7\,e^{-x/4}$$

Find the following:

(a) The optimal level of current assets. $10.72 million ♥
(b) Total annual cost of these assets at the optimal level. $1.766 million ♥

1.38. Grant Company has cost of capital 11%. It has estimated that the shortage costs are given by

$$S = 4.2\,e^{-x/10}$$

where S is the shortage cost in $million/year, and x is the amount of current assets, in $million. Find the optimal level of current assets. What is the minimum total annual cost of maintaining current assets? $13.40 million, $2.574 million ♥

1.39. Harding Company's cost of capital is 12%. It has invested x (million dollars) in current assets. The following function represents the shortage cost of current assets

$$S = 8\,e^{-x/4}$$

Find the following:

(a) The optimal level of current assets. $11.25 million ♥
(b) Total annual cost, shortage and financing, of these assets. $1.83 million ♥

Key terms