1. MATHEMATICAL PRELIMINARIES

Objectives: After reading this chapter, you should be able to
1. Solve linear and quadratic equations, and a system of two linear equations.
2. Sum geometric series, both finite and infinite.
3. Use Excel, Maple, or WolframAlpha to solve finance problems.

1.0   Khan Academy

If you need to brush up on elementary mathematics, Khan Academy is the best place to visit. They have a very large number of instructional videos at all levels of difficulty in various topics of mathematics. Besides mathematics, you will find videos on finance and on many other subjects.

1.1   WolframAlpha

WolframAlpha is free software provided by Mathematica. It is particularly useful in solving simple problems that one sees in finance. The instructions are quite intuitive and straightforward. We shall use it as a convenient way to cut through the details of calculations. If you click on this symbol WRA, you will go to WolframAlpha.

1.2   Linear Equations

To review the basic concepts of algebra, we first look at simple linear equations. These equations do not have any squares, square roots, trigonometric, or logarithmic functions. To get started, you should solve the following equations for x:

1.1. $16x - 54 = 15x - 32$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $x = 22$ ♥

1.2. $(x + 1)(x - 2) = (x - 1)(x + 2)$ $\qquad\qquad\qquad\qquad\qquad$ $x = 0$ ♥

1.3. $(10x + 3)(3x + 4) = (5x + 6)(6x + 7)$ $\qquad\qquad\qquad$ $x = -15/11$ ♥

To solve the problem on WolframAlpha, copy the instructions in blue, click on the logo, paste them, and hit the Enter key, to get the answer.

WRA   `(10x+3)(3x+4)=(5x+6)(6x+7)`

1.4. $\dfrac{x-2}{x-3} = \dfrac{x-7}{x-9}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $x = -3$ ♥

WRA `(x-2)/(x-3)=(x-7)/(x-9)`

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1.5. $\dfrac{x+4}{x+5} = \dfrac{x+6}{x+8}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $x = -2$ ♥

Next, consider sets of linear equations. We may have two equations with two unknown quantities, or three equations with three unknowns, and so on.

1.6. Solve the following equations for x and y:

$$12\,x + 6\,y = 33$$
$$17\,x + 8\,y = 46$$

First, try to eliminate one of the variables, say y. You can do this by multiplying the first equation by 8 and the second one by 6, and then subtracting the second equation from the first. This gives us

$$8{*}12\,x + 8{*}6\,y = 8{*}33$$
and $\qquad\qquad\qquad\qquad$ $6{*}17\,x + 6{*}8\,y = 6{*}46$

Or, $\qquad\qquad\qquad\qquad\qquad$ $96\,x + 48\,y = 264$
and $\qquad\qquad\qquad\qquad$ $102\,x + 48\,y = 276$

Subtracting them, $\qquad\qquad$ $(96 - 102)\,x = 264 - 276$

Simplifying it, $\qquad\qquad\qquad$ $-6\,x = -12$

Cancelling terms, $\qquad\qquad\qquad$ $x = 2$ ♥

Substituting this value of x in the first equation, we have $12{*}2 + 6\,y = 33$

Or, $\qquad\qquad\qquad\qquad\qquad$ $6y = 33 - 24 = 9$

Which gives, $\qquad\qquad\qquad\qquad$ $y = 9/6 = 3/2$ ♥

Solve the following equations for x and y:

1.7. $\qquad\qquad\qquad\qquad\qquad$ $2x + 6y = 32$
$\qquad\qquad\qquad\qquad\qquad\qquad$ $5x + 8y = 45$ $\qquad\qquad\qquad$ $x = 1, y = 5$ ♥

WRA **2x+6y=32,5x+8y=45**

1.8. $\qquad\qquad\qquad\qquad\qquad$ $3x + 4y = 15$
$\qquad\qquad\qquad\qquad\qquad\qquad$ $5x + 8y = 45$ $\qquad\qquad\qquad$ $x = -15, y = 15$ ♥

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1.3   Non-linear Equations

Non-linear equations contain higher powers of the unknown variable, or the variable itself may appear in the power of a number. Consider the following examples.

1.9. Solve for x: $1.113^x = 2.678$

First, recall the basic property of <u>logarithm</u> functions, namely,

$$\ln(a^x) = x \ln a$$

Taking the logarithm on both sides of the given equation, we obtain

$$x \ln(1.113) = \ln(2.678)$$

Or, $x = \dfrac{\ln(2.678)}{\ln(1.113)}$

To solve such problems, you need a calculator, such as TX-30, which has keys for the following functions: natural logarithm, $\boxed{\text{LN}}$, power function, $\boxed{y^x}$ and the parentheses, $\boxed{(}$ and $\boxed{)}$. In some calculators, $\boxed{y^x}$ is replaced by $\boxed{\wedge}$.

Now, press the keys in the following sequence

$$2.678 \; \boxed{\text{LN}} \; \boxed{\div} \; 1.113 \; \boxed{\text{LN}} \; \boxed{=} \; 9.201 \; \heartsuit$$

The order of keystrokes may vary depending on the model of the calculator.

$\boxed{\text{WRA}}$ `1.113^x=2.678`

1.10. Solve for x, $3.909^x = 15.99$ $x = 2.033 \; \heartsuit$

1.11. Solve for x, $(2 + x)^{2.11} = 16.55$

In this case, it is not necessary to take the logs. We may transfer the power 2.11 from the left side to the right side, which gives

$$2 + x = (16.55)^{\frac{1}{2.11}}$$

Or, $x = (16.55)^{1/2.11} - 2$

To solve the equation using a calculator equipped with the power key, $\boxed{y^x}$ and the parentheses keys, $\boxed{(}$ and $\boxed{)}$, proceed as follows:

$$16.55 \; \boxed{y^x} \; \boxed{(} \; 1 \; \boxed{\div} \; 2.11 \; \boxed{)} \; \boxed{-} \; 2 \; \boxed{=} \; 1.781 \; \heartsuit$$

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1.12. Solve for x, $(1 + x)^{3.2} = 8.4$ $x = 0.9446$ ♥

1.13. Solve for x, $1.767^x = 3.876$ $x = 2.38$ ♥

1.4   Geometric Series

In finance, we frequently have to deal with a series of payments, such as the installment payments to pay off a loan. We can solve these problems by first understanding their mathematical nature: they are geometric series. Let us consider a series of numbers represented by the following sequence

$$a,\ ax,\ ax^2,\ ax^3,\ ...,\ ax^{n-1}$$

The sequence has the property that each number, when multiplied by x, generates the next number in the list. There are altogether n terms in this series, the first one has no x, the second one has an x, the third one has x^2. By this reasoning, we know that the nth term must have x^{n-1} in it. This type of series is called a geometric series. Our concern is to find the sum of such a series having n terms with the general form

$$S = a + ax + ax^2 + ax^3 + ... + ax^{n-1} \tag{1.1}$$

To evaluate the sum, proceed as follows. Multiply each term by x and write the terms on the right side of the equation one step to the right of their original position. Set up the original and the new series as follows:

$$S = a + ax + ax^2 + ax^3 + ... + ax^{n-1}$$

$$xS = \quad ax + ax^2 + ax^3 + ... + ax^{n-1} + ax^n$$

If we subtract the second equation from the first one, most of the terms will cancel out, and we get

$$S - xS = a - ax^n$$

Or, $$S(1 - x) = a(1 - x^n)$$

or, $$S_n = \frac{a\,(1 - x^n)}{1 - x} \tag{1.2}$$

Starting with (1.1), you can get to (1.2), by using

WRA **`Sum[a*x^i,{i,0,n-1}]`**

In (1.2), we write S_n to designate the summation for n terms. This is the general expression for the summation of a geometric series with n terms, the first term being a, and the ratio between the terms being x. This is a useful formula and we can use for the summation of an annuity.

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If the number of terms in an annuity is infinite, it becomes a perpetuity. To find the sum of an infinite series, note that when n approaches infinity, $x^n = 0$ for $x < 1$. Thus, the sum for an infinite geometric series becomes

$$S_\infty = \frac{a}{1-x} \tag{1.3}$$

You may verify (1.3) as follows.

WRA **`Sum[a*x^i,{i,0,infinity}]`**

Let us use (1.2) and (1.3) to find the sum of the following geometric series:

1.14. Find the sum of $3 + 6 + 12 + 24$ ..., 13 terms

Comparing the above series with (1.1), we identify $a = 3$, $x = 2$, and $n = 13$. Putting these in (1.2), we get

$$S_n = \frac{a\,(1-x^n)}{1-x} = \frac{3\,(1-2^{13})}{1-2} = 3(2^{13} - 1) = 24{,}573 \; ♥$$

WRA **`Sum[3*2^i,{i,0,12}]`**

WRA **`Sum[3*2^(i-1),{i,1,13}]`**

1.15. Find the sum of $1.7 + 2.21 + 2.873$ ..., 11 terms

Here $a = 1.7$, and $x = 2.21/1.7 = 1.3$. Also, $n = 11$. This gives

$$S_n = \frac{a\,(1-x^n)}{1-x} = \frac{1.7\,(1-1.3^{11})}{1-1.3} = 95.89 \; ♥$$

1.16. Find the sum of $2.5 + (2.5)(.3) + (2.5)(.3)(.3)$ ..., infinite terms   3.571 ♥

A convenient way to write the summation of a series of numbers, or expressions, is to use the sigma notation, represented by the Greek letter Σ. The quantity under the Σ is the starting point of the summation index i, and the number at the top is the final value of this index. For example, one can write the series in problem (1.20) as,

$$\frac{1}{1.1} + \frac{1}{1.1^2} + \frac{1}{1.1^3} + \dots \; 9 \text{ terms} = \sum_{i=1}^{9} \frac{1}{1.1^i}$$

1.17. Find the sum of $\dfrac{1}{1.1} + \dfrac{1}{1.1^2} + \dfrac{1}{1.1^3} + \dots$ 9 terms   5.759 ♥

1.18. Evaluate $\displaystyle\sum_{i=1}^{10} \frac{500}{1.12^i}$ 2825.11 ♥

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1.5   Excel

It is important that the students are able to set up finance problems using Excel, which is now a standard of business and industry. A good working knowledge of this software should be an integral part of every student's education. Almost all business programs offer courses in the use of Excel. If you want to brush up your skill in the use of Excel, you may go the following website:

http://office.microsoft.com/en-us/training/CR100479681033.aspx

Microsoft has set up this website and it offers a variety of tutorials. Consider the following example again.

1.19. Solve for x: $1.113^x = 2.678$

	A	B	C
1	The base number =	1.113	
2	Raised to power x	9.2012	
3	Gives the result =	2.678	=B1^B2

To solve it on Excel, copy the information on an Excel sheet as above. *Leave the cell B2 blank*, because that is where the answer will appear. The formula in cell C3 will calculate the result and put it there. Start by putting some positive number, such as 3 or 7, in B2. The calculation result shows up in cell C3. By adjusting the value of the number in B2, you can make the numbers in B3 and C3 almost equal to one another. This will happen when B2 equals 9.2012. This is the unknown x.

If you are familiar with logarithms, you can write the equation as

$$x \ln(1.113) = \ln(2.678), \quad \text{or} \quad x = \ln(2.678)/\ln(1.113)$$

Now copy it on Excel as follows

	A	B
1	The base number =	1.113
2	Raised to power x	=ln(2.678)/ln(1.113)
3	Gives the result =	2.678

The answer appears in cell B2 as 9.201184226.

To solve Problem 1.17, you can set it as

	A
1	=1/1.1+1/1.1^2+1/1.1^3+1/1.1^4+1/1.1^5+1/1.1^6+1/1.1^7+1/1.1^8+1/1.1^9

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1.6   Maple

Maple is a powerful analytical tool. If you have used it before, you can review this section, otherwise you can skip it. We can use it to work out many different types of mathematical problems. It is possible to do all the above problems, with little effort, by using Maple. The help facility in Maple is extremely valuable and it can guide the user through the steps with numerous examples.

Since Maple interprets capital and lower case letters distinctly, we should use the symbols carefully. Maple has hundreds of built in mathematical functions and constants, such as

<p align="center"><code>ln, exp, Pi, sin, sqrt</code></p>

Maple can do exact arithmetic calculations and it shows the answer in totality. For example, we need the exact value of 2^{64}, or the factorial of 50, or the value of π to 50 significant figures. We may do it as follows, and striking the enter key.

`2^64`

<p align="center">18446744073709551616</p>

`50!`

<p align="center">30414093201713378043612608166064768844377641568960512000000000000</p>

`evalf(Pi,50)`

<p align="center">3.1415926535897932384626433832795028841971693993751</p>

The function **`evalf`** calculates the result in floating point with 50 significant figures.

Maple can also do algebraic calculations. For instance, to solve the equations

$$5x + 6y = 7$$
$$6x + 7y = 8$$

for *x* and *y*, we enter the instructions as follows:

`solve({5x+6y=7,6x+7y=8})`

Maple provides the answer as $\{y = 2, x = -1\}$.

Consider example 1.9 again. We can solve it on Maple as follows.

1.20. Solve for *x*: $1.113^x = 2.678$

`solve(1.113^x=2.678)`

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Maple can perform hundreds of advanced mathematical calculations. It can draw plots. It has a statistical package and a finance package, among dozens of other packages.
It is possible to add text in the plots, draw three-dimensional or animated plots, or draw plots in color. All plots in this book are drawn with the help of Maple.

To plot the familiar sine function, enter the instruction

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plot(sin(x),x=0..2*Pi)
```

Maple creates the curve shown as follows.
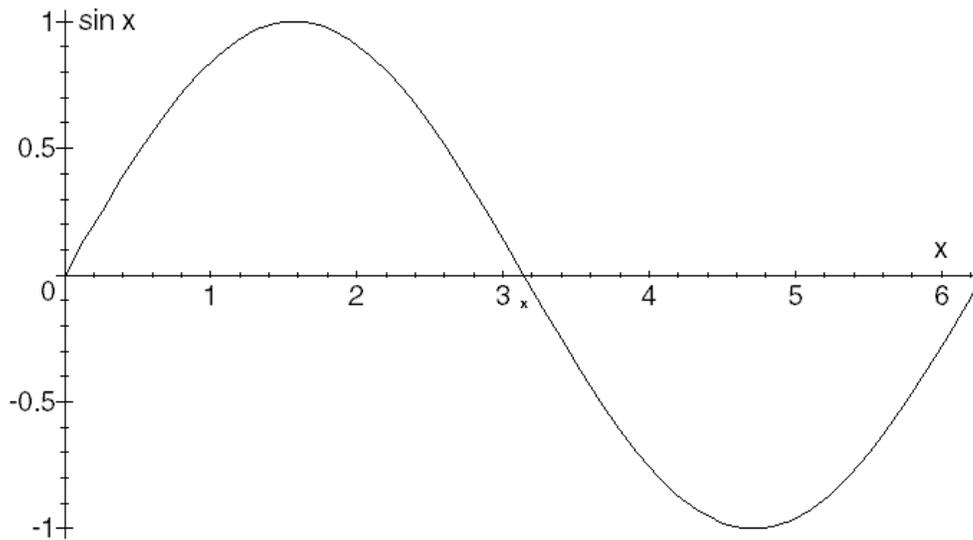

Fig. 1.1: Plot of sin x for $0 < x < 2\pi$

Maple is an extremely versatile analytical tool. It finds extensive applications in finance, mathematics, and science. Available on line, the help facility makes working with Maple exciting and fun. Any time spent in learning this program can pay rich dividends in greater accuracy and higher productivity.

1.7   Miscellaneous Problems

1.21. The Johnson family is at the ice-cream shop where they order 3 cones and one sundae. The total bill is $6. On their next visit to the same place, they had two cones and two sundaes for a total of $7. Find the price of a cone and a sundae.

<div align="right">Cone $1.25, sundae $2.25 ♥</div>

1.22. During January, the stock of Microsoft moved up by 10%; during February, it dropped by 10%; and in March it rose by 10%. If the opening price of stock on the first trading day of the year was $30 per share, find its closing price at the end of the first quarter.

<div align="right">$32.70 ♥</div>

1.23. The monthly electricity bill for PP&L customers consists of a fixed amount per month ($12.59) and a variable amount, which depends on the number of units of electricity (KWh) and the price per unit. The bill for June is $95.78 for 885 KWh, and for July, it is $105.69 for 980 KWh. Find the price of a KWh of electricity in June and in July.   June 9.4¢, July 9.5¢ ♥

HOW TO SUBMIT YOUR HOMEWORK

Consider the following problems and their solutions as a guide to the submission of your homework. Merely solving the problem is not sufficient; you have to demonstrate the mastery of the material. You need some logical thinking, the use of some financial principle, and some well-known equations to find the result. You have to *explain* your steps so that the reader can understand your work. Without proper explanations, you should not expect full credit. You should verify the result by writing an instruction that you can use at *one* of the analytical software: (a) Wolfram|Alpha, (b) Excel, or (c) Maple. This will give you the confidence that you have done something right.

1.1. Find the sum of $15.2 + 15.2/1.1 + 15.2/1.1^2 + ...$, 21 terms

This is a geometric series with $a = 15.2$, $n = 21$, and $x = 1/1.1$. To find the sum, use (1.2)

$$S_n = \frac{a\,(1 - x^n)}{1 - x}$$

Or, $$S_n = \frac{15.2(1 - 1/1.1^{21})}{1 - 1/1.1} = 144.61 \; ♥$$

You can check the answer at WolframAlpha by typing

WRA `Sum[15.2/1.1^i,{i,0,20}]`

The solution consists of the following steps.

1. Identify the problem as a geometric series.
2. Identify the data and write it in symbols.
3. Use an equation in the textbook to solve the problem.
4. Show the numbers in the equation properly.
5. Calculate the result.
6. Verify the result by using (a) WolframAlpha, (b) Excel, or (c) Maple

1.2. During January, the stock of Microsoft moved up by 10%; during February, it dropped by 10%; and in March it rose by 10%. If the opening price of stock on the first trading day of the year was $30 per share, find its closing price at the end of the first quarter.

The stock price rose 10% in January and it became $30(1.1) = \$33$ at the end of January.

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It dropped by 10% in February, and it was 33(.9) = $29.70
It rose again 10% in March and it became 29.70(1.1) = $32.67 ♥

Since it increased $2.67, starting with $30, the percentage change is 2.67/30 = .089 = 8.9% ♥

You can check the answer at WolframAlpha by typing

WRA `30*1.1*.9*1.1`

WRA `1.1*.9*1.1-1`

The solution consists of the following steps

1. Use logical steps to show the final stock price after three months.
2. Use some logic to find the percentage change in the price.
3. Verify the result by using (a) WolframAlpha, (b) Excel, or (c) Maple

1.3. In January 1985, the price of gold was $300 per oz. In January 2011, it was $1400 per oz. Find the increase in the price of gold per year, compounded annually, during this period.

Use equation (2.1),
$$FV = PV\,(1 + r)^n$$

Put $PV = 300$, $FV = 1400$, $n = 2011 - 1985 = 26$, and r is the unknown number. This gives
$$1400 = 300(1 + r)^{26}$$

$$1400/300 = (1 + r)^{26}$$

$$(14/3)^{1/26} = 1 + r$$

$$r = (14/3)^{1/26} - 1 = .061038225 = 6.104\% \; ♥$$

The check the result, use the following instruction at Wolfram|Alpha.

WRA `1400=300(1+r)^26`

The solution consists of the following steps.

1. Identify an equation in the textbook that can solve the problem.
2. Identify the numbers and symbols in the equation.
3. Show the steps to solve the problem.
4. Verify the result by using (a) WolframAlpha, (b) Excel, or (c) Maple

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1.4. Suppose you put \$250 at the beginning of every month in a savings account that credits interest at the annual rate of 6%, but compounds it monthly. Find the amount in this account after 25 years.

The monthly rate of interest is ½%, or 0.005. Consider the first deposit of \$250. Its future value after 25 years (or 300 months) is $250(1.005)^{300}$. The second deposit is a month late; it has only 299 months to grow, and its final value is $250(1.005)^{299}$. In a similar way, we find that the last deposit has just one month to earn interest. Putting it all together, the following expression gives the total at the end of 300 months:

$$S = 250(1.005)^{300} + 250(1.005)^{299} + ... + 250(1.005)$$

This is a geometric series with $a = 250(1.005)^{300}$, and $n = 300$. The exponent of the factor (1.005) is decreasing. This implies that the multiplicative factor $x = 1/1.005$. Using (1.2),

$$S_n = \frac{a\,(1 - x^n)}{1 - x} \tag{1.2}$$

we find

$$FV = \frac{250(1.005)^{300}(1 - 1/1.005^{300})}{1 - 1/1.005} = \$174,114.73 \text{ ♥}$$

You can verify the answer at <u>WolframAlpha</u> as follows.

WRA `Sum[250*1.005^i,{i,1,300}]`

The solution consists of the following steps.

1. Differentiate between annual and monthly interest rates.
2. Using logical steps, convert the word problem to a mathematical expression.
3. Identify an equation in the textbook that can solve the problem.
4. Identify the numbers and symbols in the equation.
5. Show the steps to solve the problem.
6. Verify the result by using (a) <u>WolframAlpha</u>, (b) Excel, or (c) Maple

Presenting your work on Excel

These are some general suggestions to improve the quality of your work when you present it using Excel.

1. Do only one problem on a sheet.

2. Some persons have difficulty reading across a long line. Notice that the newspaper columns are rather narrow, even though the page is packed with print. To make the Excel files easier to read, do the following.

Try to use columns A-F if possible.
The verbal explanations should be approximately 6 inches across.

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Avoid having too much blank space on the Excel sheet.

3. Ideally, your work should have the appearance as if it is on a 8½ x 11 inch sheet. Make sure the print is easy to read – not too large, not too small.

In general, I find it difficult to follow the work in Excel, compared to the same material presented in a Word file.

Key Terms

Differentiation	Integration	Quadratic equation
Excel	Linear equations	Sigma notation
Geometric series	Maple	Summation index
Infinite series	Perpetuity	