

## 2. ANALYTICAL TOOLS

**Goals:** After reading this chapter, you will

1. Know the basic concepts of statistics: expected value, standard deviation, variance, covariance, and coefficient of correlation.
2. Use Maple effectively to solve portfolio related problems.

### 2.1 [Video 01B](#), Statistical Quantities

Before we can study portfolio theory in earnest, it is desirable to develop some useful tools that will make the study a little easier. The subject of [statistics](#) plays a vital role in describing the uncertain outcomes of events. For example, we may not know the exact return of an investment, but we can say something about its [expected return](#). We can say that the expected return of Ford stock is 15%, but we are not sure about it. To assess the accuracy of our estimate we can use another parameter called the [standard deviation](#) of returns.

Let us briefly review some of the fundamental definitions from statistics that we plan to use in portfolio theory. First, we define the [expected value](#) of a [random variable](#)  $X$ . Let us describe a random variable  $X$  by a [discrete probability distribution](#)  $P_i$ , where  $i = 1..n$ . We then we define

$$\text{Expected value of } X, \quad E(X) = \sum_{i=1}^n P_i X_i = \bar{X} \quad (2.1)$$

In other words, we multiply each outcome with the probability of that outcome and then sum all the products. This will give us the expected value. We must first develop a subjective probability distribution that would describe the random event. The sum of all the probabilities is one.

Another useful quantity is the [variance](#) of a random variable, meaning the dispersion, or scatter in its value. We define it as

$$\text{Variance of } X, \quad \text{var}(X) = \sum_{i=1}^n P_i (X_i - \bar{X})^2 \quad (2.2)$$

The square root of the variance is the [standard deviation](#), which describes the scatter, or margin of error of a random variable. The advantage of using the standard deviation as a measure of dispersion is that it has the same units as the random variable. We define it as

$$\text{Standard deviation of } X, \quad \sigma_X = \sqrt{\text{var}(X)} \quad (2.3)$$

Standard deviation is a measure of the error in our estimate of the expected value of an uncertain event. If we know the outcome of an event with absolute certainty, then its standard deviation, or error in the estimate, is zero.

Quite often, two random events are interrelated. The value of one depends on the other. For instance, the return of one bank stock may depend upon the return of another bank stock. This is because the entire banking industry may benefit from lower interest rates, or lower reserve requirements of the Federal Reserve.

The interdependence of two random variables, or their interaction, is expressed in terms of their [covariance](#), defined as

$$\text{Covariance between } X \text{ and } Y, \quad \text{cov}(X, Y) = \sum_{i=1}^n P_i (X_i - \bar{X})(Y_i - \bar{Y}) \quad (2.4)$$

The quantity covariance is difficult to use in practice. A more practical parameter is the quantity  $r_{XY}$ , the [correlation coefficient](#) between  $X$  and  $Y$ , which is defined as

$$r_{XY} = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} \quad (2.5)$$

Or, 
$$\text{cov}(X, Y) = \sigma_X \sigma_Y r_{XY} \quad (2.6)$$

There are at least two advantages of using the correlation coefficient. First, it is a dimensionless number, and second, its value lies between +1 and -1. That is

$$-1 < r_{XY} < 1 \quad (2.7)$$

The correlation coefficient between two random variables measures their interdependence. A strong linkage between them will result in a correlation coefficient close to 1. If the correlation coefficient is exactly 1, then the two variables are perfectly positively correlated. For correlation coefficient close to zero, the two variables are quite independent of each other. If the two variables are completely negatively correlated, the correlation coefficient between them will be -1.

## 2.2 Continuous Probability

Consider a standardized test, such as SAT. The test scores of a large number of candidates will tend to show a certain pattern. The scores will tend to bunch around the mean score, and fall off on both sides. The [normal probability distribution](#) function can describe this pattern in an approximate way. If the mean score is  $\mu$  and the standard deviation of the scores is  $\sigma$ , then the normal probability distribution function,  $P(x)$  is

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad (2.8)$$

For  $\sigma = 1$  and  $\mu = 0$ , it simplifies to the standard normal distribution,  $n(x)$

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad (2.9)$$

An arbitrary normal distribution becomes a standard normal distribution by changing variables to  $z = (x - \mu)/\sigma$ , and  $dz = dx/\sigma$ . Thus

$$n(x) dx = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \quad (2.10)$$

The cumulative normal distribution function  $N(d)$  gives the probability that a standard normal variate assumes a value in the interval  $[-\infty, d]$ , where

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-z^2/2} dz \quad (2.11)$$

One can calculate the value of  $N(d)$  by using the table in Chapter 13. The plot of  $n(x)$  for  $-3 < x < 3$  is in the following diagram.

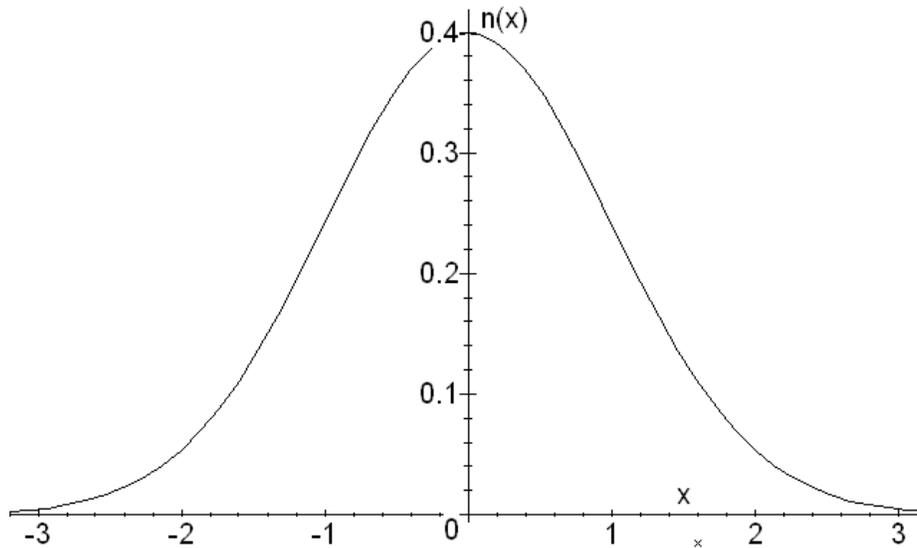


Figure 2.1: The normal probability density function  $n(x)$  as defined by (2.9).

To understand these concepts, consider the following example.

### Example

**2.1.** A financial analyst has developed the following data about the state of the economy and the returns of two stocks.

State of the Economy	Probability	Return on GM	Return on Ford
Good	50%	35%	30%
Average	30%	10%	5%
Poor	20%	-30%	-25%

Find: (a) The expected return of both stocks.  
 (b) The standard deviation of the stocks.  
 (c) The correlation coefficient between GM and Ford.

(a) The expected value of the return for each stock is

$$\text{GM: } E(R) = .5*.35 + .3*.1 - .2*.3 = .145 = 14.5\%$$

$$\text{Ford: } E(R) = .5*.3 + .3*.05 - .2*.25 = .115 = 11.5\%$$

(b) We find the standard deviation as

$$\text{GM: } \sigma(R) = \sqrt{.5(.35 - .145)^2 + .3(.1 - .145)^2 + .2(-.3 - .145)^2} = .2474$$

$$\text{Ford: } \sigma(R) = \sqrt{.5(.3 - .115)^2 + .3(.05 - .115)^2 + .2(-.25 - .115)^2} = .2122$$

The somewhat higher  $\sigma$  of GM implies that there is greater uncertainty in the returns of this stock.

(c) Next, we calculate the covariance between the stocks. We do it as

$$\begin{aligned} \text{cov}(G,F) &= .5(.35 - .145)(.3 - .115) + .3(.1 - .145)(.05 - .115) + .2(-.3 - .145)(-.25 - .115) \\ &= .052325 \end{aligned}$$

From (2.6), we have

$$r_{GF} = \frac{\text{cov}(G,F)}{\sigma_G \sigma_F} = \frac{.052325}{.2474 * .2122} = .9967$$

The extremely high value of correlation coefficient, which is nearly 1, says that the two companies are almost carbon copies of one another. The impact of the economic conditions on the two companies is almost identical.

### 2.3 Excel

It is important that the students are able to set up finance problems using Excel, which is now a standard of business and industry. A good working knowledge of this software should be an integral part of every business student's education. Almost all business programs offer courses in the use of this software. If you want to brush up your skill in the use of Excel, you may go the following Microsoft website for a variety of tutorials.

<http://office.microsoft.com/en-us/training/CR100479681033.aspx>

To get started on Excel, consider one of the previous problems that we solved by using the logarithm function.

**1.2.** Solve for  $x$ :  $1.113^x = 2.678$

Set up the table shown below. Adjust the number in the **green** cell B2 until the numbers in cells B3 and B4 come very close together. B2 gives the answer.

	A	B
1	Base =	1.113
2	Unknown power =	9.201184226
3	Result (given) =	2.678
4	Result(calculated) =	=B1^B2

It is possible to embed an Excel table within a Word document. To do that, go the **Insert** tab in a Word document. When it opens, click on **Table**. In the **Table** menu, click on **Excel Spreadsheet** near the bottom. An Excel sheet opens up, where you can do your work. When you finish your Excel work, click anywhere on the Word document, and you can leave Excel. To go back into the Excel spreadsheet, double-click on the table, which will reveal all the calculations and formulas.

Next, consider example 2.1 on page 7 again. Set it up on Excel as follows. The numerical results of the formulas in cells B5:B10 are given in **green** cells C5:C10. The principal advantage of Excel is that it can handle large tables of numbers.

	A	B	C	D
1	State of the Economy	Probability	Return on GM	Return on Ford
2	Good	60%	35%	30%
3	Fair	30%	10%	5%
4	Poor	10%	-30%	-25%
5	E(G)	=B2*C2+B3*C3+B4*C4	0.145	
6	E(F)	=B2*D2+B3*D3+B4*D4	0.115	
7	Cov(G,F)	=B2*(C2-B5)*(D2-B6)+B3*(C3-B5)*(D3-B6)+B4*(C4-B5)*(D4-B6)	0.052325	
8	sigma(G)	=SQRT(B2*(C2-B5)^2+B3*(C3-B5)^2+B4*(C4-B5)^2)	0.247436861	
9	sigma(F)	=SQRT(B2*(D2-B6)^2+B3*(D3-B6)^2+B4*(D4-B6)^2)	0.212190952	
10	r(G,F)	=B7/B8/B9	0.996593337	

#### 2.4 [Video 01C](#) Maple

[Maple](#) is a powerful computer software that can do complex mathematical calculations. Working with Maple is quite easy. You simply turn the computer on, click on the Maple button, and you are ready to work. The help facility is extremely valuable and it can guide the user through different steps with illustrative examples. It makes working with Maple exciting and fun. Maple is an extremely versatile analytical tool. It is used extensively in science, mathematics, engineering, and finance. Any time spent in learning this program can pay rich dividends in greater accuracy and higher productivity. The following instructions should get you started in the use of Maple.

Since Maple interprets capital and lower case letters distinctly, we should use the symbols carefully. Maple has many built in mathematical functions and constants, such as

**ln, exp, Pi, sin, sqrt**

Maple can do exact arithmetic calculations and displays the answer in its totality. For example, we need the exact value of  $2^{64}$ , or the factorial of 50, or the value of  $\pi$  to 50 significant figures. We do this as follows: enter the commands at the  $>$  prompt, end each line with a semicolon, and strike the return key.

```
2^64;
```

```
18446744073709551616
```

```
50!;
```

```
30414093201713378043612608166064768844377641568960512000000000000
```

```
evalf(Pi,50);
```

```
3.1415926535897932384626433832795028841971693993751
```

Here

```
evalf
```

calculates the result in floating point with 50 significant figures. Maple can also do algebraic calculations. For instance, to solve the equations

$$5x + 6y = 7$$

$$6x + 7y = 8$$

for  $x$  and  $y$ , we enter the instructions as follows:

```
eq1:=5*x+6*y=7;
```

```
eq1 := 5 x + 6 y = 7
```

```
eq2:=6*x+7*y=8;
```

```
eq2 := 6 x + 7 y = 8
```

```
solve({eq1,eq2},{x,y});
```

```
{y = 2, x = -1}
```

The symbol

```
:=
```

is used specifically to *define* objects in Maple. In other words, if we type in

```
eq1;
```

then the computer will recall the equation defined as `eq1` and display it as

$$5 x + 6 y = 7$$

Maple can also do differentiation and integration. Consider the function

$$x^3 + \frac{\ln x}{x}$$

To differentiate this function with respect to  $x$ , we type in

```
diff(x^3+ln(x)/x,x);
```

$$3x^2 + \frac{1}{x^2} - \frac{\ln(x)}{x^2}$$

To integrate the result with respect to  $x$ , recreating the original function, we enter

```
int(% ,x);
```

$$x^3 + \frac{\ln x}{x}$$

Here we use

```
%
```

as a symbol to designate the previous expression.

We can also use Maple to plot functions. For instance, if we want to see the visual representation of the well-known sine wave, we write

```
plot(sin(x),x=0..2*Pi);
```

which gives the diagram shown below.

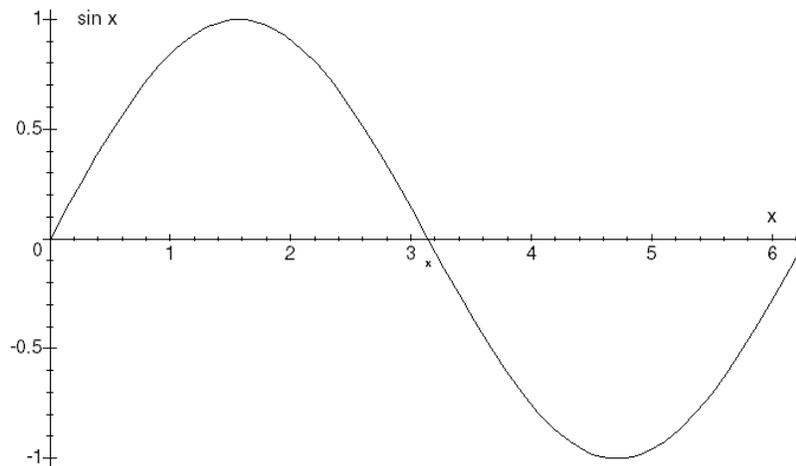


Fig. 2.2: Plot of  $\sin x$  for  $0 < x < 2\pi$

It is possible to add text in the plots, draw three-dimensional or animated plots, or draw plots in color. All plots in this book are drawn with the help of Maple.

## 2.5 Wolfram|Alpha

[Mathematica](#) is another analytical software, which has capabilities similar to Maple. It can perform all the mathematical problems equally well. Mathematica has a website at [Wolfram|Alpha](#), which is free to use. The instructions at Wolfram|Alpha are almost identical to those in Maple. You should explore this website and use it when you do not have access to Maple.

For instance, to solve the equations

$$\begin{aligned} 5x + 6y &= 7 \\ 6x + 7y &= 8 \end{aligned}$$

for  $x$  and  $y$ , enter the instructions as follows:

**5x+6y=7 , 6x+7y=8**

When you click on the = sign, it provides the solution as  $x = -1$ ,  $y = 2$ .

To see the sine wave of Figure 1.1, write

**Plot[Sin[x] , {x, 0, 2Pi}]**

### Example

**2.2.** A portfolio made of stocks of Oslo Company and Quito Company has  $E(R_p) = 12\%$  and  $\beta_p = 1.3$ . The  $\beta$  of Oslo is 1.4 and that of Quito 1.1. The risk-free rate is 8%. Find the expected return on the market, and the weights of the two stocks in the portfolio.

To set it up on Maple, we proceed as follows.

```
ERp := .12;
betap := 1.3;
beta1 := 1.4;
beta2 := 1.1;
RF := .08;
eq1 := ERp = w1*ER1 + w2*ER2;
eq2 := betap = w1*beta1 + w2*beta2;
eq3 := ERp = RF + betap*(ERm - RF);
eq4 := ER1 = RF + beta1*(ERm - RF);
eq5 := ER2 = RF + beta2*(ERm - RF);
solve({eq1, eq2, eq3, eq4, eq5}, {w1, w2, ER1, ER2, ERm});
```

The desired result is

$w_1 = .6667$ ,  $w_2 = .3333$ ,  $E(R_1) = 12.31\%$ ,  $E(R_2) = 11.38\%$ , and  $E(R_m) = 11.08\%$ . ♥

## Problems

2.3. You have developed the following data about the state of the economy and the returns of two stocks.

State of the Economy	Probability	Return on Dell	Return on Intel
Good	50%	25%	35%
Average	30%	5%	5%
Poor	20%	-35%	0%

Find: (a) The expected return of both the stocks. 7%, 19% ♥  
 (b) The standard deviation of the stocks. 22.72%, 16.09% ♥  
 (c) The correlation coefficient between Dell and Intel 85.35% ♥

2.4. Write a set of Maple instructions to solve problem 2.3.

```
n:=3;
P:=array(1..n, [.5,.3,.2]);
R1:=array(1..n, [.25,.05,-.35]);
R2:=array(1..n, [.35,.05,0]);
ER1:=sum(P[i]*R1[i],i=1..n);
ER2:=sum(P[i]*R2[i],i=1..n);
SD1:=sqrt(sum(P[i]*(R1[i]-ER1)^2,i=1..n));
SD2:=sqrt(sum(P[i]*(R2[i]-ER2)^2,i=1..n));
Cov12:=sum(P[i]*(R1[i]-ER1)*(R2[i]-ER2),i=1..n);
r12:=Cov12/SD1/SD2;
```

Solve the following equations with the help of Maple:

2.5.  $16x - 54 = 15x - 32$   $x = 22$  ♥

2.6.  $(x + 1)(x - 2) = (x - 1)(x + 2)$   $x = 0$  ♥

2.7.  $(10x + 3)(3x + 4) = (5x + 6)(6x + 7)$   $x = -15/11$  ♥

2.8.  $\frac{x-2}{x-3} = \frac{x-7}{x-9}$   $x = -3$  ♥

2.9.  $\frac{x+4}{x+5} = \frac{x+6}{x+8}$   $x = -2$  ♥

2.10.  $2x + 6y = 32$   
 $5x + 8y = 45$   $x = 1, y = 5$  ♥

2.11.  $3x + 4y = 15$   
 $5x + 8y = 45$   $x = -15, y = 15$  ♥

---

2.12.	$(1 + x)^{3.2} = 8.4$	$x = 0.9446$ ♥
2.13.	$1.767^x = 3.876$	$x = 2.38$ ♥
2.14.	$3.909^x = 15.99$	$x = 2.033$ ♥
2.15.	$2x^2 + 7x - 9 = 0$	$x = 1, -9/2$ ♥
2.16.	$3x^2 + 4x - 7 = 0$	$x = 1, -7/3$ ♥

**Multiple-Choice Question**

1. An example of a correct Maple instruction is

- A. `z:=ln(a*x)/x^4`
- B. `U:=ln(5W)/7-exp(-W);`
- C. `plot(y^3/4+y^2/sin(y),y=1..2;`
- D. `solve({5*x+y=6,x*y+y=2},{x,y});`